A family of rational maps with buried Julia components

Sébastien Godillon

April 16, 2013

Abstract

It is known that the disconnected Julia set of any polynomial map does not contain buried Julia components. But such Julia components may arise for rational maps. The first example is due to Curtis T. McMullen who provided a family of rational maps for which the Julia sets are Cantor of Jordan curves. However all known examples of buried Julia components, up to now, are points or Jordan curves and comes from rational maps of degree at least 5.

This paper introduce a family of hyperbolic rational maps with disconnected Julia set whose exchanging dynamics of critically separating Julia components is encoded by a weighted Hubbard tree. Each of these Julia sets presents buried Julia components of several types: points, Jordan curves, but also Julia components which are neither points nor Jordan curves. Moreover this family contains some rational maps of degree 3 with explicit formula that answers a question McMullen raised.

Contents

1 Introduction

For any rational map f of degree $d \geqslant 2$ on the Riemann sphere $\widehat{\mathbb{C}}$, we denote by $J(f)$ its Julia set, namely the closure of the set of repelling periodic points. We recall that $J(f)$ is a fully invariant non-empty perfect compact set which either is connected or has uncountably many connected components (see [Bea91], [CG93], [Mil06]). This paper focuses on the disconnected case. Every connected component of $J(f)$ is called a Julia component and every connected component of the Fatou set $\widehat{\mathbb{C}} - J(f)$ is called a Fatou domain. We denote by by $\mathcal{J}(f)$ the set of Julia components and we recall that f induces a dynamical system on $\mathcal{J}(f)$, called the exchanging dynamics.

A Julia component is said to be **buried** if it has no intersection with the boundary of any Fatou domain. In particular buried Julia components can not occur in the polynomial case (since the Julia set coincides with the boundary of the unbounded Fatou domain). The same holds if the Julia set is a Cantor set, or more generally if the complementary of every Julia component is connected (since the Fatou set is then connected). That suggests sophisticated topological structure for Julia sets with some buried Julia components.

The first example of rational maps with buried Julia components is due to Curtis T. McMullen. Consider the family of rational maps given by

$$g_{c,\lambda} : z \mapsto z^{d_\infty} + c + \frac{\lambda}{z^{d_0}} \quad \text{where } d_\infty, d_0 \geqslant 1 \text{ and } c, \lambda \in \mathbb{C}$$

The special case $c = 0$ has been studied in [McM88] (see also [DHL+08]), where it is proved that if the following condition is satisfied

$$\frac{1}{d_\infty} + \frac{1}{d_0} < 1 \tag{H0}$$

and if $|\lambda| > 0$ is small enough then $J(g_{0,\lambda})$ is a Cantor of Jordan curves, namely homeomorphic to the product of a Cantor set with a Jordan curve (see Figure 1). Recall that any Cantor set is homeomorphic to the no-middle third set $[0,1] \setminus \cup_{n \geqslant 1} \cup_{k=0}^{3^{n-1}-1} I_{n,k}$ where $I_{n,k}$ is the real open interval $]\frac{3k+1}{3^n}, \frac{3k+2}{3^n}[$. Remark that the no-middle third set contains uncountably many points which are not endpoints of any segment $I_{n,k}$ and each of these points corresponds to a buried Jordan curve in $J(g_{0,\lambda})$.

In [PT00], the authors has provided another example by slightly modifying the map $g_{-1,\lambda}$ for $d_\infty = 2$ and $d_0 = 3$ (that satisfies assumption (H0)) in a clever way:

$$\widetilde{g_{-1,\lambda}} : z \mapsto \frac{1}{z} \circ (z^2 - 1) \circ \frac{1}{z} + \frac{\lambda}{z^3} = \frac{z^2}{1 - z^2} + \frac{\lambda}{z^3} \quad \text{where } \lambda \in \mathbb{C}$$

If $|\lambda| > 0$ is small enough then $J(\widetilde{g_{-1,\lambda}})$ has the same topological structure than $J(g_{0,\lambda})$ except that one fixed Julia component (which contains the boundary of the unbounded Fatou domain and hence is not buried) is quasiconformally homeomorphic to the Julia set of $z \mapsto z^2 - 1$. The uncountably many Julia components which are not eventually mapped under iterations onto this fixed Julia component are buried Jordan curves in $J(\widetilde{g_{-1,\lambda}})$ (see Figure 1).

Examples of buried Jordan components which are not Jordan curves have appeared in some works. For instance in [BDGR08] (see also [DM08] and [GMR13]), the authors have studied the family $g_{c,\lambda}$ for $d_\infty = d_0 \geqslant 3$ (that satisfies assumption (H0)) and for a fixed parameter c chosen so that for the polynomial $z \mapsto z^{d_\infty} + c$ the critical point 0 lies in a cycle of period at least 2. In that case, if $|\lambda| > 0$ is small enough then $J(g_{c,\lambda})$ still has uncountably many Jordan curves as buried Jordan components but also uncountably many

points. The remaining Julia components are eventually mapped under iterations onto a fixed Julia component (which coincides with the boundary of the unbounded Fatou domain and hence is not buried) quasiconformally homeomorphic to the Julia set of $z \mapsto z^{d_\infty} + c$. Each of these not buried Julia components has infinitely many "decorations" and every buried point component is actually the accumulation point of a nested sequence of such decorations (see Figure 1).

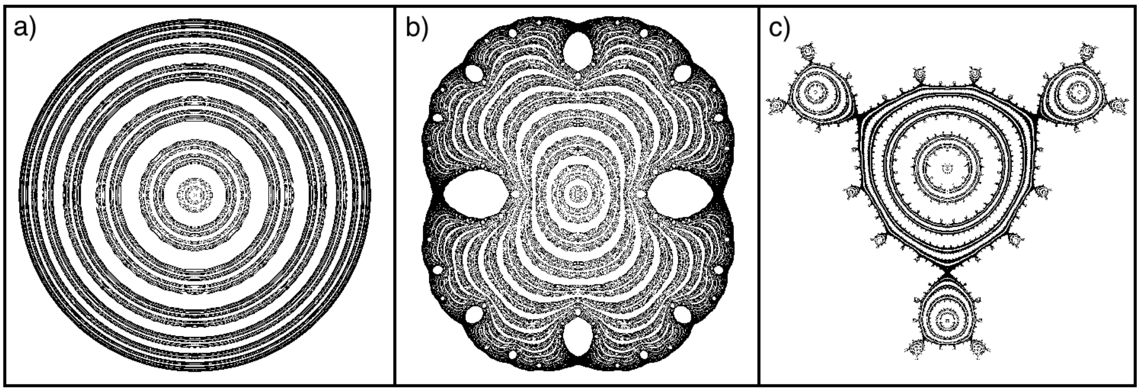


FIGURE 1: **a)** $J(g_{0,\lambda})$ with $d_\infty = 2$, $d_0 = 3$ and $\lambda \approx 10^{-9}$
b) $J(\widetilde{g_{-1,\lambda}})$ with $d_\infty = 2$, $d_0 = 3$ and $\lambda \approx 10^{-9}$
c) $J(g_{c,\lambda})$ with $d_\infty = d_0 = 3$, $c = -i$ and $\lambda \approx 10^{-9}$

Remark that each of the previous examples are rational maps of degree $d_\infty + d_0$ at least 5 according to assumption (H0). The existence question of buried Julia components for rational maps of degree less than 5 has been raised in [McM88]. In the last decade, a number of papers have appeared that deal with subfamilies of $g_{c,\lambda}$ or some slightly perturbations of it. Some of them present sophisticated Julia sets with buried Julia components, however the degree of these examples is always at least equal to 5. Furthermore the buried Julia components of these examples are points or Jordan curves.

The aim of this paper is to answer the question Curtis T. McMullen has raised by providing a family of rational maps of degree 3 which does not come from the family $g_{c,\lambda}$ and whose Julia set presents buried Julia components of several types: points, Jordan curves but also Julia components which are neither points nor Jordan curves. One of our main result here is the following

Theorem 1. *Consider the family of cubic rational maps given by*

$$f_\lambda : z \mapsto \frac{(1-\lambda)\Big[(1-4\lambda+6\lambda^2-\lambda^3)z - 2\lambda^3\Big]}{(z-1)^2\Big[(1-\lambda-\lambda^2)z - 2\lambda^2(1-\lambda)\Big]} \quad \textit{where } \lambda \in \mathbb{C}$$

If $|\lambda| > 0$ is small enough then $J(f_\lambda)$ contains buried Julia components of several types:

point type: *uncountably many points*

circle type: *uncountably many Jordan curves*

complex type: *countably many preimages of a fixed Julia component which is quasiconformally homeomorphic to the connected Julia set of $f_0 : z \mapsto \frac{1}{(z-1)^2}$*

An example of such Julia set is depicted in Figure 2. $J(f_\lambda)$ is called a "Persian carpet" because of similarities with sophistications from carpet-weaving art: the Julia set of $f_0 : z \mapsto \frac{1}{(z-1)^2}$ appears as a watermark in the central motif of the carpet whose surface is covered by an elaborate pattern of Cantor of Jordan curves, and there are some small Julia components everywhere that looks like dust. These small Julia components actually contain nested sequences of finite coverings of the Persian carpet which accumulate buried point components.

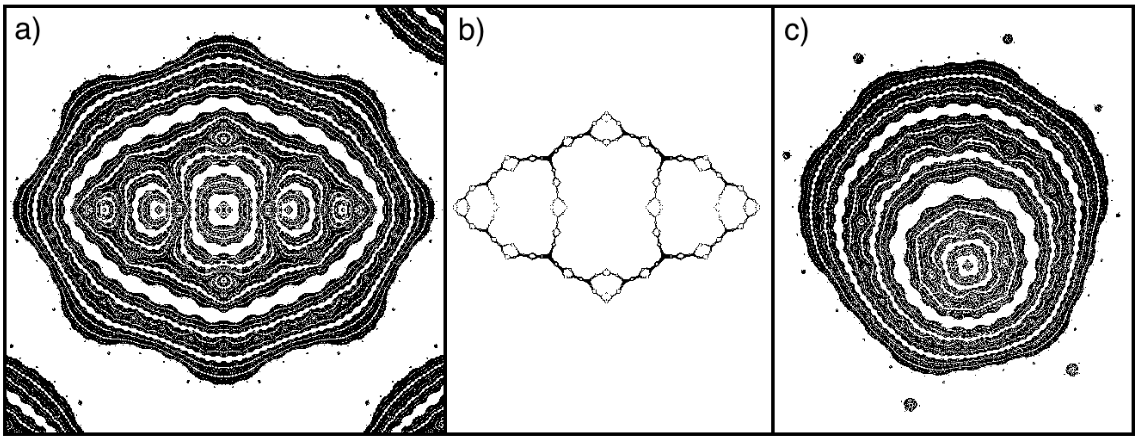


FIGURE 2: **a)** A Persian carpet: $J(f_\lambda)$ with $\lambda \approx 10^{-3}$
b) $J(f_0)$ which appears as a buried Julia component in $J(f_\lambda)$
c) A magnification about a dust of the Persian carpet

The Persian carpet example is maximal among rational maps with buried Julia components in the sense that buried Julia components can not occur for rational maps of degree less than 3. Indeed, by a theorem in [Mil00], the Julia set of any quadratic rational maps is either connected or a Cantor set.

Furthermore, the Persian carpet example is maximal among geometrically finite rational maps (namely rational maps such that every critical point in the Julia set is preperiodic, in our case f_λ has no critical point in $J(f_\lambda)$ for $|\lambda| > 0$ small enough) in the sense that every Julia component (not necessarily buried) of such a map is one of the three types described in Theorem 1. That follows from two results. Firstly, by a theorem in [McM88], every periodic Julia component of a rational map is either a point or quasiconformally homeomorphic to the connected Julia set of a rational map. Secondly, it has been proved in [PT00] that every wandering Julia component (namely a Julia component which is not eventually mapped under iterations onto a periodic Julia component) of a geometrically finite rational map is either a point or a Jordan curve.

The main idea of this paper is that the exchanging dynamics of critically separating Julia components (a Julia component is said to be critically separating if its complementary has at least two connected components containing some postcritical point) for some rational maps may be encoded by weighted Hubbard trees. This idea is specified in Section 2.1 by showing that, under assumption (H0), the exchanging dynamics for the family $g_{0,\lambda}$ is topologically conjugated to that one coming from a Hubbard tree $\mathcal{H}_Q$ (see Theorem 2). Weights are added on the edges of $\mathcal{H}_Q$ in order to carry information about the degrees d_∞, d_0 of the restrictions of $g_{0,\lambda}$ to each Julia component.

In Section 2.2 the purpose is then to do the converse: starting from a Hubbard tree $\mathcal{H}_P$ more sophisticated than $\mathcal{H}_Q$ and a weight function w on its edges, Theorem 3 states

the existence of rational maps with disconnected Julia set whose exchanging dynamics of critically separating Julia components is encoded by $(\mathcal{H}_P, w)$ if (and, actually, only if) the weight function w satisfies two conditions (H1) and (H2). Theorem 4 shows that the Julia sets of these rational maps own buried Julia components of every type as explained above.

The main part of the proof of Theorem 3, that is the construction by quasiconformal surgery of the required rational maps, is detailed in Section 3.

In Section 4, some properties of the rational maps constructed in the previous section are shown. The properties about exchanging dynamics (Section 4.1) conclude the proof of Theorem 3 while the properties about topology of some Julia components (Section 4.2) give the proof of Theorem 4.

Finally Section 5 deals with a particular choice of the weight function w for which the two assumptions (H1) and (H2) are satisfied and such that the rational maps in Theorem 3 and Theorem 4 are of degree 3. In this case, an explicit formula is provided that concludes the proof of Theorem 1.

Acknowledgment. The author would like very much to thank Professor Tan Lei, the advisor of his thesis, together with Professor Cui Guizhen for all their helpful comments and fruitful discussions on this work.

2 Encoding by weighted Hubbard trees

2.1 McMullen example

Consider the cubic polynomial $Q : z \mapsto 3z^2(\frac{3}{2} - z)$. It has two simple critical points: 0 which is fixed and 1 which is mapped on 0 after two iterations.

$$\begin{tikzcd} 0 \arrow[loop left, "2:1"] & 1 \arrow[r, "2:1"] & \frac{3}{2} \arrow[ll, bend right, "1:1"'] \end{tikzcd}$$

Let $\mathcal{H}_Q$ be its Hubbard tree, namely the smallest closed connected infinite union of internal rays which contains the postcritical set $\{0, \frac{3}{2}\}$ (see [DH84]). In fact $\mathcal{H}_Q$ is the straight real segment $[0, \frac{3}{2}]$ or more precisely the union of two edges $[0, 1] \cup [1, \frac{3}{2}]$ while the vertices are 0, 1 and $\frac{3}{2}$. Both edges of $\mathcal{H}_Q$ are homeomorphically mapped by Q onto the whole tree (see Figure 3).

Denote by $\mathcal{J}(\mathcal{H}_Q)$ the intersection set between the Hubbard tree $\mathcal{H}_Q$ and the Julia set $J(Q)$. Notice that $\mathcal{J}(\mathcal{H}_Q)$ is disconnected (actually a Cantor set) and Q induced a dynamical system on it since the Hubbard tree $\mathcal{H}_Q$ and the Julia set $J(Q)$ are both invariant.

Finally, let w be a weight function on the set of edges of $\mathcal{H}_Q$, say $w([0, 1]) = d_\infty$ and $w([1, \frac{3}{2}]) = d_0$ where d_∞, d_0 are positive integer.

The result about the family $g_{0,\lambda}$ discussed in introduction (see Section 1) may be reformulated as follows.

Theorem 2. *If the weight function w satisfies the following condition*

$$\frac{1}{d_\infty} + \frac{1}{d_0} < 1 \tag{H0}$$

then for every $|\lambda| > 0$ small enough, the dynamical system induced by $g_{0,\lambda}$ on the set of its Julia components $\mathcal{J}(g_{0,\lambda})$ is encoded by the weighted Hubbard tree $(\mathcal{H}_Q, w)$ in the following sense

(i) *every critical orbit accumulates the super-attracting fixed point* ∞

(ii) *there exists a homeomorphism* $h : \mathcal{J}(g_{0,\lambda}) \to \mathcal{J}(\mathcal{H}_Q)$ *such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{J}(g_{0,\lambda}) \arrow[r, "g_{0,\lambda}"] \arrow[d, "h"'] & \mathcal{J}(g_{0,\lambda}) \arrow[d, "h"] \\ \mathcal{J}(\mathcal{H}_Q) \arrow[r, "Q"'] & \mathcal{J}(\mathcal{H}_Q) \end{tikzcd}$$

(iii) *for every Julia component* $J \in \mathcal{J}(g_{0,\lambda})$, *the restriction map* $g_{0,\lambda}|_J$ *is of degree* $w(e)$ *where* e *is the edge of* $\mathcal{H}_Q$ *which contains* $h(J)$

Proof. We only sketch the proof since the main part is done in [McM88]. Indeed it is shown that there exists a large annulus A centered at 0 and containing $J(g_{0,\lambda})$ whose preimage consists of two disjoint annuli A_∞, A_0 both nested in A and such that the restriction maps $g_{0,\lambda}|A_\infty : A_\infty \to A$ and $g_{0,\lambda}|A_0 : A_0 \to A$ are coverings of degree d_∞, d_0 respectively. Using combinatorial reasoning from complex dynamics, it is a classical exercise to prove that the set of connected components of $J(g_{0,\lambda}) = \cap_{n\geqslant 0} g_{0,\lambda}^{-n}(A)$ is homeomorphic to the space of all sequences of two digits $\Sigma_2 = \{0,1\}^{\mathbb{N}}$ (equipped with the product topology making it a Cantor set) and the exchanging dynamics is topologically conjugated to a 2-to-1 shift map $\sigma : \Sigma_2 \to \Sigma_2$ defined by $\sigma(s_0, s_1, s_2, \dots) = (s_1, s_2, s_3, \dots)$. The same holds for the dynamical system induced by Q on $\mathcal{J}(\mathcal{H}_Q)$ since for $\varepsilon > 0$ small enough the real segment $I = [\varepsilon, \frac{3}{2} - \varepsilon]$ contains $\mathcal{J}(\mathcal{H}_Q)$ and its preimage consists of two disjoint real segment both included in I (one in each of the two edges of $\mathcal{H}_Q$). $\square$

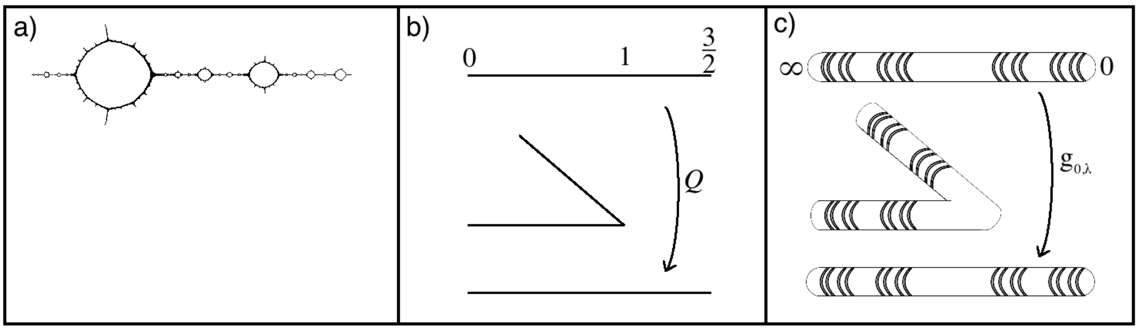
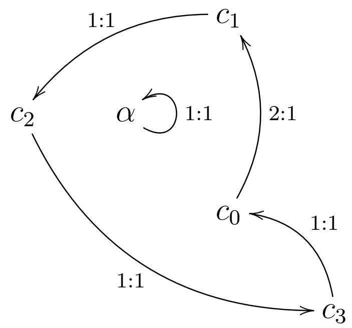


FIGURE 3: **a)** The Julia set of the polynomial Q
b) The action of Q on the Hubbard tree $\mathcal{H}_Q$
c) The action of $g_{0,\lambda}$ on the set of Julia components $\mathcal{J}(g_{0,\lambda})$

Heuristically speaking, we may topologically think the Riemann sphere $\widehat{\mathbb{C}}$ as a smooth neighborhood's boundary of the Hubbard tree $\mathcal{H}_Q$ embedded in the space $\mathbb{R}^3$. The two points on this topological sphere which correspond to ∞ and 0 should be closed to the corresponding vertices of $\mathcal{H}_Q$, that is 0 and $\frac{3}{2}$ respectively. If the neighborhood becomes smaller and smaller, every Jordan curves in $J(g_{0,\lambda})$ is shrinked to a point in $J(\mathcal{H}_Q)$. Moreover the dynamical tree $Q : \mathcal{H}_Q \to \mathcal{H}_Q$ together with the weight function w encode the action of $g_{0,\lambda}$ on the neighborhood's boundary with respect to this heuristic (see Figure 3).

2.2 Persian carpets example

Consider a quadratic polynomial of the form $P : z \mapsto z^2 + c$ where the parameter $c \in \mathbb{C}$ is chosen in order that the critical point 0 is periodic of period 4. There are exactly six choices of such a parameter. Let us fix c to be that one with the largest imaginary part, that is $c \approx -0.157 + 1.032i$. The postcritical points are denoted by $c_k = P^k(0)$ for every $k \in \{0, 1, 2, 3\}$.



Let $\mathcal{H}_P$ be the Hubbard tree of P (see Figure 4). As one-dimensional simplicial complex, $\mathcal{H}_P$ may be described by a set of five vertices $\{c_0, c_1, c_2, c_3, \alpha\}$ where α is a fixed point of P and the following four edges

$$e_0 = [\alpha, c_0]_{\mathcal{H}_P},\ e_1 = [\alpha, c_1]_{\mathcal{H}_P},\ e_2 = [\alpha, c_2]_{\mathcal{H}_P},\ e_{3=}[c_0, c_3]_{\mathcal{H}_P}$$

P homeomorphically acts on the edges as follows

$$\begin{cases} P(e_0) = e_1 \\ P(e_1) = e_2 \\ P(e_2) = e_0 \cup e_3 \\ P(e_3) = e_0 \cup e_1 \end{cases}$$

Denote by $\mathcal{J}(\mathcal{H}_P)$ the intersection set between the Hubbard tree $\mathcal{H}_P$ and the Julia set $J(P)$. Notice that $\mathcal{J}(\mathcal{H}_P)$ is disconnected (actually a Cantor set) and P induced a dynamical system on it. Moreover the fixed branching point α belongs to $\mathcal{J}(\mathcal{H}_P)$ but not to the boundary of any connected component of $\mathcal{H}_P - \mathcal{J}(\mathcal{H}_P)$. Finally, let w be a weight function on the set of edges of $\mathcal{H}_P$, say $w(e_k) = d_k$ where d_k is a positive integer for every $k \in \{0, 1, 2, 3\}$.

Definition 1. The transition matrix of the weighted Hubbard tree $(\mathcal{H}_P, w)$ is the 4-square matrix $M = (m_{i,j})_{i,j \in \{0,1,2,3\}}$ whose entries are defined as follows

$$\forall i, j \in \{0, 1, 2, 3\},\ m_{i,j} = \begin{cases} \dfrac{1}{w(e_i)} & \text{if } e_j \subset P(e_i) \\ 0 & \text{otherwise} \end{cases}$$

Since M is a non-negative matrix, it follows from Perron-Frobenius theorem that the eigenvalue with the largest modulus is real and non-negative. Let us call $\lambda(\mathcal{H}_P, w)$ this leading eigenvalue. The weighted Hubbard tree $(\mathcal{H}_P, w)$ is said to be **unobstructed** if $\lambda(\mathcal{H}_P, w) < 1$.

Let us give some remarks about this definition.

1. This definition is strongly related to obstructions which occur in Thurston characterization of postcritically finite rational maps and all the theory behind.

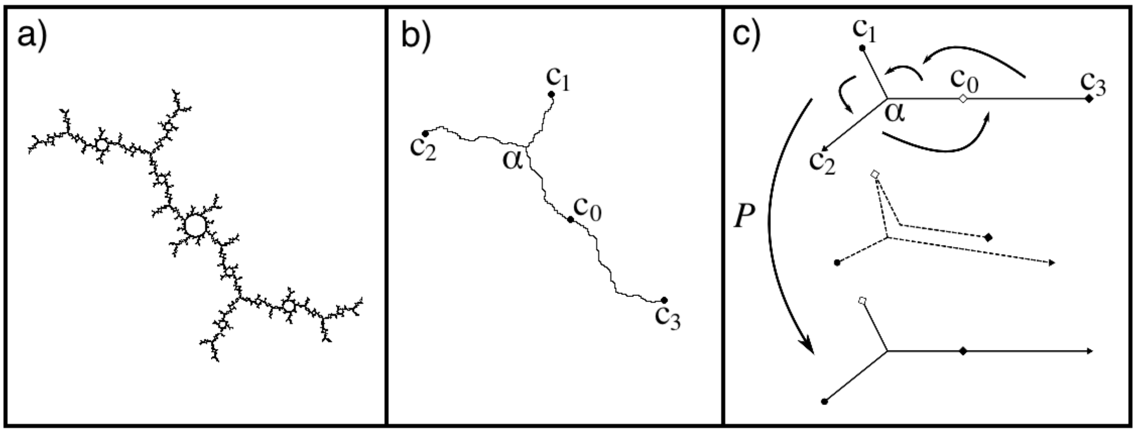


FIGURE 4: **a)** The Julia set of the polynomial P
b) The Hubbard tree $\mathcal{H}_P$
c) The action of P on a straightened copy of $\mathcal{H}_P$

2. When $(\mathcal{H}_P, w)$ is unobstructed, Perron-Frobenius theorem and continuity of the spectral radius ensure the existence of a vector $V \in \mathbb{R}^4$ with positive entries such that $MV < V$. This remark will be useful later.

3. Actually the transition matrix of $(\mathcal{H}_P, w)$ is given by

$$M = \begin{pmatrix} 0 & \frac{1}{d_0} & 0 & 0 \\ 0 & 0 & \frac{1}{d_1} & 0 \\ \frac{1}{d_2} & 0 & 0 & \frac{1}{d_2} \\ \frac{1}{d_3} & \frac{1}{d_3} & 0 & 0 \end{pmatrix}$$

and an easy computation shows that $\lambda(\mathcal{H}_P, w)$ is the largest root of

$$X^4 - \left(\frac{1}{d_0 d_1 d_2} + \frac{1}{d_1 d_2 d_3}\right) X - \frac{1}{d_0 d_1 d_2 d_3}$$

Notice that if $\lambda(\mathcal{H}_P, w) \geqslant 1$ then $\lambda(\mathcal{H}_P, w) \leqslant \frac{1}{d_0 d_1 d_2} + \frac{1}{d_1 d_2 d_3} + \frac{1}{d_0 d_1 d_2 d_3}$. Consequently $(\mathcal{H}_P, w)$ is unobstructed as soon as at least three of weights d_0, d_1, d_2, d_3 are $\geqslant 2$.

4. For the McMullen example, the transition matrix of $(\mathcal{H}_Q, w)$ may be defined as well and we get

$$M = \begin{pmatrix} \frac{1}{d_\infty} & \frac{1}{d_\infty} \\ \frac{1}{d_0} & \frac{1}{d_0} \end{pmatrix}$$

An easy computation gives that $\lambda(\mathcal{H}_Q, w) = \frac{1}{d_\infty} + \frac{1}{d_0}$. Consequently the weighted Hubbard tree $(\mathcal{H}_Q, w)$ is unobstructed if and only if the assumption (H0) holds.

Finally, recall that a Julia component $J \in \mathcal{J}(f)$ of a rational map f is said to be critically separating if its complementary has at least two connected components containing some postcritical points, namely some iterates of critical points. We denote by $\mathcal{J}_{\text{crit}}(f)$ the set of critically separating Julia components. Remark that every Julia component of $g_{0,\lambda}$ for every $|\lambda| > 0$ small enough is critically separating and hence $\mathcal{J}(g_{0,\lambda}) = \mathcal{J}_{\text{crit}}(g_{0,\lambda})$.

We can now state the main result of this paper by analogy with Theorem 2.

Theorem 3. *If the weight function w satisfies the two following conditions*

$$\widehat{d} = \frac{1}{2}(d_0 + d_1 + d_2 - 1) \text{ is an integer} \geqslant 2 \text{ and } \max\{d_0, d_1, d_2\} \leqslant \widehat{d} \tag{H1}$$

$$(\mathcal{H}_P, w) \text{ is unobstructed} \tag{H2}$$

then there exists a rational map f of degree $\widehat{d} + d_3$ with disconnected Julia set such that the dynamical system induced by f on the set of its critically separating Julia components $\mathcal{J}_{crit}(f)$ is encoded by the weighted Hubbard tree $(\mathcal{H}_P, w)$ in the following sense

(i) *every critical orbit accumulates a super-attracting cycle $\{z_0, z_1, z_2, z_3\}$ of period 4*

(ii) *there exists a homeomorphism $h : \mathcal{J}_{crit}(f) \to \mathcal{J}(\mathcal{H}_P)$ such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{J}_{crit}(f) \arrow[r, "f"] \arrow[d, "h"'] & \mathcal{J}_{crit}(f) \arrow[d, "h"] \\ \mathcal{J}(\mathcal{H}_P) \arrow[r, "P"'] & \mathcal{J}(\mathcal{H}_P) \end{tikzcd}$$

(iii) *for every Julia component $J \in \mathcal{J}_{crit}(f)$ such that $h(J)$ is not eventually mapped under iteration to the fixed branching point α, the restriction map $f|_J$ is of degree $w(e_k) = d_k$ where e_k is the edge of $\mathcal{H}_P$ which contains $h(J)$*

The same heuristic as for Theorem 2 still holds: we may topologically think the Riemann sphere $\widehat{\mathbb{C}}$ as a smooth neighborhood's boundary of the Hubbard tree $\mathcal{H}_P$ embedded in the space $\mathbb{R}^3$. The action of f on this topological sphere follows that one of the dynamical tree $P : \mathcal{H}_P \to \mathcal{H}_P$. The points on this topological sphere which correspond to the points in the super-attracting periodic cycle $\{z_0, z_1, z_2, z_3\}$ should be closed to the corresponding vertices $\{c_0, c_1, c_2, c_3\}$ of $\mathcal{H}_P$. And every Julia component in $\mathcal{J}_{\text{crit}}(f)$ closely surrounds a corresponding point in $\mathcal{J}(\mathcal{H}_P)$.

The next result deals with buried Julia components of f.

Theorem 4. *Under assumptions (H1) and (H2) there exists a rational map f satisfying Theorem 3 and such that $J(f)$ contains buried Julia components of several types:*

point type: *uncountably many points*

circle type: *uncountably many Jordan curves*

complex type: *countably many preimages of a fixed Julia component over the fixed branching point α, say $J_\alpha = h^{-1}(\alpha) \in \mathcal{J}(f)$, which is quasiconformally homeomorphic to the Julia set of a rational map $\widehat{f}$*

Moreover $\widehat{f}$ is of degree $\widehat{d}$ and has only one critical orbit which is a super-attracting cycle $\{\widehat{z_0}, \widehat{z_1}, \widehat{z_2}\}$ of period 3 such that the local degree of $\widehat{f}$ at $\widehat{z_k}$ is d_k for every $k \in \{0, 1, 2\}$. In particular $J(\widehat{f})$ is connected and the Fatou set $\widehat{\mathbb{C}} - J(\widehat{f})$ has infinitely many connected components.

Let us give some comments about these results.

1. The rational map f is not postcritically finite since $J(f)$ is disconnected (but it is hyperbolic from point **(i)** in Theorem 3). In particular Thurston characterization of postcritically finite rational maps is not allowed to prove the existence of f. However these results are strongly related to the works of Tan Lei and Cui Guizhen about sub-hyperbolic semi-rational maps in [CT11].

2. The rational map f is not unique since the critical points which do not belong to the super-attracting periodic cycle $\{z_0, z_1, z_2, z_3\}$ (but whose orbits accumulate it) may be perturbed in some neighborhoods without changing the exchanging dynamics and the topology of Julia components.

3. The rational map $\widehat{f}$ is unique up to conjugation by a Möbius map or equivalently it is unique as soon as its critical orbit $\{\widehat{z_0}, \widehat{z_1}, \widehat{z_2}\}$ is fixed in $\widehat{\mathbb{C}}$ (see Lemma 2).

4. The assumption (H1) is necessary. Indeed it is the smallest requirement such that there exists a topological model for $\widehat{f}$, that is an orientation-preserving branched covering combinatorially equivalent to $\widehat{f}$ (see Lemma 1 and proof of Lemma 2).

5. The assumption (H2) is necessary. Otherwise we can find a Thurston obstruction, that is to say a multicurve Γ whose transition matrix is equal to M with leading eigenvalue $\lambda(\Gamma) = \lambda(\mathcal{H}_P, w) \geqslant 1$. According to a result of Curtis T. McMullen from [McM94] it follows that $\lambda(\Gamma) = 1$ and at least one curve in Γ is contained in an union of Fatou domains where f is biholomorphically conjugated to a rotation. That is a contradiction since every critical orbit of f accumulates a super-attracting periodic cycle.

6. The rational map $\widehat{f}$ may also be seen as encoded by a weighted Hubbard tree. Consider the quadratic polynomial $R : z \mapsto z^2 + \widehat{c}$ where $\widehat{c} \in \mathbb{C}$ is the parameter with the largest imaginary part such that the critical point 0 is periodic of period 3, that is $\widehat{c} \approx -0.123 + 0.745i$. The Hubbard tree $\mathcal{H}_R$ of R is described by a set of four vertices $\{\widehat{c_0}, \widehat{c_1}, \widehat{c_2}, \widehat{\alpha}\}$ where $\widehat{c_k} = R^k(0)$ and $\widehat{\alpha}$ is a fixed point of R, and three edges of the form $\widehat{e_k} = [\widehat{\alpha}, \widehat{c_k}]_{\mathcal{H}_R}$ for every $k \in \{0, 1, 2\}$. Consider the weight function w defined by $w(\widehat{e_k}) = d_k$ for every $k \in \{0, 1, 2\}$. Then the weighted Hubbard tree $(\mathcal{H}_R, w)$ encodes the action of $\widehat{f}$ in the same setting as in Theorem 2 and Theorem 3. Notice that the intersection set between $\mathcal{H}_R$ and $J(R)$ is reduced to $\mathcal{J}(\mathcal{H}_R) = \{\widehat{\alpha}\}$, that corresponds to the unique Julia component in $\mathcal{J}(\widehat{f}) = \mathcal{J}_{\text{crit}}(\widehat{f}) = \{J(\widehat{f})\}$. Finally, remark that the weighted Hubbard tree $(\mathcal{H}_R, w)$ is unobstructed as soon as (H1) holds (actually $\lambda(\mathcal{H}_R, w) = \frac{1}{d_0 d_1 d_2}$).

3 Construction

The aim of this section is to construct by quasiconformal surgery the resulting map f satisfying Theorem 3 and Theorem 4. The strategy is to start from a rational map $\widehat{f}$ whose Julia set corresponds to the branching point α in $\mathcal{H}_P$ (see Theorem 4) and then to modify this map in order to create a folding corresponding to the critical point c_0 (see Figure 4).

3.1 Existence of the branching map $\widehat{f}$

The first step of the construction is to prove the existence of the rational map $\widehat{f}$ which appears in Theorem 4. This is done by Lemma 2 below. Lemma 1 is a general preliminary about the existence of a topological model for $\widehat{f}$.

Lemma 1 (Branched covering extension)**.** *Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite subset of $n \geqslant 2$ points in the topological sphere $\mathbb{S}^2$. Let $F : X \to X$ be a bijective map and $d_1, d_2, \ldots, d_n$ be a family of positive integers not all equal to 1. Then F can be extended to an orientation-preserving branched covering $\widehat{F} : \mathbb{S}^2 \to \mathbb{S}^2$ with local degree d_k at x_k for every $k \in \{1, 2, \ldots, n\}$ and no other critical points outside X if and only if*

$$d = \frac{1}{2}\sum_{k=1}^{n}(d_k - 1) + 1 \textit{ is an integer } \geqslant 2 \textit{ and } \max\{d_1, d_2, \ldots, d_n\} \leqslant d \tag{H1'}$$

In that case, the extended map $\widehat{F} : \mathbb{S}^2 \to \mathbb{S}^2$ is of degree d.

Proof. The necessity of (H1') easily comes from classical properties of branched coverings. In order to proof the sufficiency, let us prove the following claim.

Claim: Let $\delta_1, \delta_2, \ldots, \delta_m$ be a family of $m \geqslant 1$ positive integers. Assume that

$$\delta = \sum_{k=1}^{m}(\delta_k - 1) + 1 \text{ is an integer } \geqslant 2 \text{ and } \max\{\delta_1, \delta_2, \ldots, \delta_m\} \leqslant \delta \tag{H1''}$$

Then there exist a continuous map $G : \overline{U} \to \overline{V}$ between closures of two Jordan domains U, V and two finite subsets $\{u_1, u_2, \ldots, u_m\} \subset U$ and $\{v_1, v_2, \ldots, v_m\} \subset V$ such that

- $G|_U : U \to V$ is an orientation-preserving branched covering of degree δ such that $G(u_k) = v_k$ and the local degree of G at u_k is δ_k for every $k \in \{1, 2, \ldots, m\}$
- $G|_{\partial U} : \partial U \to \partial V$ is homeomorphically equivalent to the map $z \mapsto z^\delta$ on the unit circle $\{z \in \mathbb{C} \,/\, |z| = 1\}$

Proof of the claim. Let $\{u_1, u_2, \ldots, u_m\}$ be a finite subset in the Riemann sphere $\widehat{\mathbb{C}}$. Let $G : \mathbb{C} \to \mathbb{C}$ be any antiderivative of the polynomial map $z \mapsto \prod_{k=1}^{m}(z - u_k)^{\delta_k - 1}$. Then G is a polynomial map of degree δ (according to (H1'')) such that the local degree of G at u_k is δ_k for every $k \in \{1, 2, \ldots, m\}$. Denote by v_k the image of u_k under G for every $k \in \{1, 2, \ldots, m\}$. Remark that each v_k polynomially depends on the points $u_1, u_2, \ldots, u_m$ and these polynomial relations are distinct. Consequently we may carefully choose the distinct points $u_1, u_2, \ldots, u_n$ in order that $v_1, v_2, \ldots, v_n$ are distinct as well. The result follows with $U = \{z \in \mathbb{C}, / \,|z| < R\}$ and $V = G(U)$ for $R > 0$ large enough. $\square$

Now come back to the proof of Lemma 1. Up to postcomposition with an orientation-preserving homeomorphism on $\mathbb{S}^2$, we may assume that F is the identity map on X.

First case: $\exists \ell \in \{1, 2, \ldots, n-1\} / \sum_{k=1}^{\ell}(d_k - 1) + 1 = \sum_{k=\ell+1}^{n}(d_k - 1) + 1 = d$

Let S be a copy of $\widehat{\mathbb{C}}$ seen as the topological sphere $\mathbb{S}^2$. Denote by $C = \{z \in S \,/\, |z| = 1\}$ the corresponding unit circle and by $D^0 = \{z \in S \,/\, |z| < 1\}$ and $D^\infty = \{z \in S \,/\, |z| > 1\} \cup \{\infty\}$ the two connected components of $S - C$. Without loss of generality we may assume that $\{x_1, x_2, \ldots, x_\ell\} \subset D^0$ and $\{x_{\ell+1}, x_{\ell+2}, \ldots, x_n\} \subset D^\infty$.

We are going to piecewisely define a continuous map $\widehat{F} : S \to S$ according to this partition of S. At first define $\widehat{F}$ on C to be the map $z \mapsto z^d$, in particular $\widehat{F}(C) = C$.

Let $G^0 : U^0 \to V^0$ be an orientation-preserving branched covering of degree d coming from the claim for the family of integers $d_1, d_2, \ldots, d_\ell$. Let $\varphi^0 : U^0 \to D^0$ and $\psi^0 : V^0 \to D^0$ be two homeomorphisms such that $\varphi^0(u_k^0) = x_k = \psi^0(v_k^0)$ for every $\forall k \in \{1, 2, \ldots, \ell\}$. Moreover φ^0 and ψ^0 may be carefully chosen in order that $\psi^0 \circ G^0 \circ (\varphi^0)^{-1} : D^0 \mapsto D^0$ can be continuously extended with the map $z \mapsto z^d$ on $\partial D^0 = C$. So define $\widehat{F}$ on D^0 to be the map $\psi^0 \circ G^0 \circ (\varphi^0)^{-1}$.

Do as well on D^∞ for the family of integers $d_{\ell+1}, d_{\ell+2}, \dots, d_n$. We get an orientation-preserving branched covering $G^\infty : U^\infty \to V^\infty$ of degree d and two homeomorphisms $\varphi^\infty : U^\infty \to D^\infty$ and $\psi^\infty : V^\infty \to D^\infty$ such that $\varphi^\infty(u_k^\infty) = x_{\ell+k} = \psi^\infty(v_k^\infty)$ for every $k \in \{1, 2, \dots, n-\ell\}$ and $\psi^\infty \circ G^\infty \circ (\varphi^\infty)^{-1} : D^\infty \mapsto D^\infty$ can be continuously extended with the map $z \mapsto z^d$ on $\partial D^\infty = C$. So define $\widehat{F}$ on D^∞ to be the map $\psi^\infty \circ G^\infty \circ (\varphi^\infty)^{-1}$.

Consequently we have well defined a continuous map $\widehat{F} : S \to S$ which is actually a branched covering of degree d. Moreover, according to the previous construction, $\widehat{F}$ satisfies every requirement from Lemma 1.

Second case: $\exists \ell \in \{2, 3, \dots, n-1\} / \sum_{k=1}^{\ell-1}(d_k - 1) + 1 < d < \sum_{k=1}^{\ell}(d_k - 1) + 1$
Then we can find two integers $d_\ell^0, d_\ell^\infty \geqslant 2$ such that

$$\sum_{k=1}^{\ell-1}(d_k - 1) + (d_\ell^0 - 1) + 1 = \sum_{k=\ell+1}^{n}(d_k - 1) + (d_\ell^\infty - 1) + 1 = d \tag{1}$$

Let S be a copy of $\widehat{\mathbb{C}}$ seen as the topological sphere $\mathbb{S}^2$. Denote by C, D^0 and D^∞ the same subsets as in the first case. Denote by $R_\theta^0 = \{re^{i\frac{2\pi}{d}\theta} / 0 \leqslant r \leqslant 1\}$ and $R_\theta^\infty = \{re^{i\frac{2\pi}{d}\theta} / 1 \leqslant r\} \cup \{\infty\}$ the closed straight rays of angle $\frac{2\pi}{d}\theta \in \mathbb{R}$ in D^0 and D^∞ respectively. Let T be the following union of rays

$$T = \left(\bigcup_{k=0}^{d_\ell^0-1} R_k^0\right) \cup \left(\bigcup_{k=0}^{d_\ell^\infty-1} R_{d-k}^\infty\right)$$

It follows from (1) that $d_\ell^0 + d_\ell^\infty = d_\ell + 1 \leqslant d + 1$ and thus $d_\ell^0 - 1 < d - d_\ell^\infty + 1$. Consequently T is connected and contains no loop (equivalently $S - T$ is connected and simply connected). Denote the connected components of $S - (C \cup T)$ as follows

$$\begin{aligned}
\forall k \in \{0, 1, \dots, d_\ell^0 - 2\}, \quad A_k^0 &= \left\{re^{i\frac{2\pi}{d}\theta} / 0 < r < 1 \text{ and } k < \theta < k+1\right\} \\
B^0 &= \left\{re^{i\frac{2\pi}{d}\theta} / 0 < r < 1 \text{ and } d_\ell^0 - 1 < \theta < d\right\} \\
\forall k \in \{0, 1, \dots, d_\ell^\infty - 2\}, \quad A_k^\infty &= \left\{re^{i\frac{2\pi}{d}\theta} / 0 < r < 1 \text{ and } d-k-1 < \theta < d-k\right\} \\
B^\infty &= \left\{re^{i\frac{2\pi}{d}\theta} / 0 < r < 1 \text{ and } 0 < \theta < d - d_\ell^\infty + 1\right\}
\end{aligned}$$

Without loss of generality, we may assume that $\{x_1, x_2, \dots, x_{\ell-1}\} \subset B^0$ and $\{x_{\ell+1}, x_{\ell+2}, \dots, x_n\} \subset B^\infty$.

We are going to piecewisely define a continuous map $H : S \to S$ according to this partition of S. At first define H on $C \cup T$ to be the map $z \mapsto z^d$, in particular $H(C \cup T) = C \cup (R_0^0 \cup R_0^\infty)$.

For every $k \in \{0, 1, \dots, d_\ell^0 - 2\}$, remark that ∂A_k^0 is a Jordan curve and $H(\partial A_k^0) = C \cup R_0^0$. More precisely, $\partial A_k^0 = C_k \cup R_k^0 \cup R_{k+1}^0$ where C_k is the closed arc $\{e^{i\frac{2\pi}{d}\theta} / k \leqslant \theta \leqslant k+1\}$, the map $H|_{C_k} : C_k \to C$ is a continuous surjective map of degree 1 and the maps $H|_{R_k^0} : R_k^0 \to R_0^0$ and $H|_{R_{k+1}^0} : R_{k+1}^0 \to R_0^0$ are homeomorphisms. It follows that H can be continuously extended to a homeomorphism $H|_{A_k^0} : A_k^0 \to D^0 - R_0^0$.

Similarly, H can be extended to a homeomorphism $H|_{A_k^\infty} : A_k^\infty \to D^\infty - R_0^\infty$ for every $k \in \{0, 1, \dots, d_\ell^\infty - 2\}$.

Now remark that ∂B^0 is a Jordan curve and $H(\partial B^0) = C \cup R_0^0$. More precisely, $\partial B^0 = C^0 \cup R_{d_\ell^0-1}^0 \cup R_0^0$ where C^0 is the closed arc $\{e^{i\frac{2\pi}{d}\theta} / d_\ell^0 - 1 \leqslant \theta \leqslant d\}$, the map $H|_{C^0} : C^0 \to C$ is a continuous surjective map of degree $d - d_\ell^0 + 1$ and the maps $H|_{R_{d_\ell^0-1}^0} : R_{d_\ell^0-1}^0 \to R_0^0$ and $H|_{R_0^0} : R_0^0 \to R_0^0$ are homeomorphisms.

Let $G^0 : U^0 \to V^0$ be an orientation-preserving branched covering coming from the claim for the family of integers $d_1, d_2, \ldots, d_{\ell-1}$. It follows from (1) that G^0 is of degree $d - d_\ell^0 + 1$. Choose a continuous path R_{V^0} in $V^0 - \{v_1^0, v_2^0, \ldots, v_{\ell-1}^0\}$ with one endpoint in V^0 and the other one in ∂V^0. Choose also a connected component of the preimage under G^0 of R_{V^0}. This preimage is a continuous path in $U^0 - \{u_1^0, u_2^0, \ldots, u_{\ell-1}^0\}$, say R_{U^0}, with one endpoint in U^0 and the other one in ∂U^0. Remark that $G^0|_{R_{U^0}} : R_{U^0} \to R_{V^0}$ is a homeomorphism.

Let $\varphi^0 : U^0 - R_{U^0} \to B^0$ and $\psi^0 : V^0 - R_{V^0} \to D^0 - R_0^0$ be two homeomorphisms such that $\varphi^0(u_k^0) = x_k = \psi^0(v_k^0)$ for every $\forall k \in \{1, 2, \ldots, \ell - 1\}$. It follows from everything above that φ^0 and ψ^0 may be carefully chosen in order that $\psi^0 \circ G^0 \circ (\varphi^0)^{-1} : B^0 \mapsto D^0 - R_0^0$ can be continuously extended with the map H on ∂B^0. Finally define H on B^0 to be the map $\psi^0 \circ G^0 \circ (\varphi^0)^{-1}$.

Do as well on B^∞ for the family of integers $d_{\ell+1}, d_{\ell+2}, \ldots, d_n$. So H can be continuously extended to a map $H : B^\infty \to D^\infty - R_0^\infty$ such that $H(x_k) = x_k$ and the local degree of H at x_k is d_k for every $k \in \{\ell + 1, \ell + 2, \ldots, n\}$.

Consequently we have well defined a continuous map $H : S \to S$ which is actually a branched covering of degree d such that

- for every $k \in \{1, \ldots, \ell - 1, \ell + 1, \ldots n\}$, $H(x_k) = x_k$ and the local degree of H at x_k is d_k
- $H(0) = 0$, $H(\infty) = \infty$ and the local degree of H at 0 (respectively at ∞) is $d^0\ell$ (respectively d_ℓ^∞)
- H has no other critical points outside $\{x_1, \ldots, x_{\ell-1}, x_{\ell+1}, \ldots, x_n\} \cup \{0, \infty\}$

Remark that $T' = H^{-1}(R_0^0 \cup R_0^\infty)$ is the union of T with some $\varphi^0(R'_{U^0})$ where R'_{U^0} describe the set of connected components of $(G^0)^{-1}(R_{V^0})$ distinct from R_{U^0} and with some $\varphi^\infty(R'_{U^\infty})$ where R'_{U^∞} describe the set of connected components of $(G^\infty)^{-1}(R_{V^\infty})$ distinct from R_{U^∞}. In particular, T' is connected and contains no loop (equivalently $S - T'$ is connected and simply connected). Moreover $H|_{T'} : T' \to R_0^0 \cup R_0^\infty$ is a continuous surjective map of degree $d_\ell^0 + d_\ell^\infty - 1$ which is equal to d_ℓ from (1).

Finally let $\varphi : S - T' \to \mathbb{S}^2 - \{x_\ell\}$ and $\psi : S - (R_0^0 \cup R_0^\infty) \to \mathbb{S}^2 - \{x_\ell\}$ be two homeomorphisms such that $\varphi(x_k) = x_k = \psi(x_k)$ for every $k \in \{1, \ldots, \ell - 1, \ell + 1, \ldots n\}$. Then the map $\widehat{F} = \psi \circ H \circ \varphi^{-1} : \mathbb{S}^2 - \{x_\ell\} \to \mathbb{S}^2 - \{x_\ell\}$ can be continuously extended to $\mathbb{S}^2$ with $\widehat{F}(x_\ell) = x_\ell$. The extended map $\widehat{F} : \mathbb{S}^2 \to \mathbb{S}^2$ is actually an orientation-preserving branched covering of degree d whose local degree at x_ℓ is d_ℓ. According to the previous construction, $\widehat{F}$ satisfies every requirement from Lemma 1. □

Lemma 2. *Let d_0, d_1, d_2 be three positive integers. Assume that*

$$\widehat{d} = \frac{1}{2}(d_0 + d_1 + d_2 - 1) \text{ is an integer} \geqslant 2 \text{ and } \max\{d_0, d_1, d_2\} \leqslant \widehat{d} \tag{H1}$$

Then there exists a rational map $\widehat{f}$ of degree $\widehat{d}$ such that

(i) *$\widehat{f}$ has only one critical orbit which is a super-attracting cycle $\{\widehat{z}_0, \widehat{z}_1, \widehat{z}_2\}$ of period 3 such that the local degree of $\widehat{f}$ at $\widehat{z}_k$ is d_k for every $k \in \{0, 1, 2\}$*

(ii) *$J(\widehat{f})$ is connected and the Fatou set $\widehat{\mathbb{C}} - J(\widehat{f})$ has infinitely many connected components which are simply connected.*

Furthermore $\widehat{f}$ is unique up to conjugation by a Möbius map.

Proof. Up to conjugation by a Möbius map, we may fix a subset $\{\widehat{z_0}, \widehat{z_1}, \widehat{z_2}\} \subset \widehat{\mathbb{C}}$. Let $\psi : \mathbb{S}^2 \to \widehat{\mathbb{C}}$ be a homeomorphism and denote by x_k the preimage under ψ of $\widehat{z_k}$ for every $k \in \{0, 1, 2\}$. Notice that assumption (H1) ensures that at least two of integers d_0, d_1, d_2 are $\geqslant 2$. Apply Lemma 1 to the cyclic permutation F on $X = \{x_0, x_1, x_2\}$ defined by $F(x_0) = x_1$, $F(x_1) = x_2$ and $F(x_2) = x_0$. We get an orientation-preserving branched covering $\widehat{F}$ of degree $\widehat{d}$ such that every critical point belongs to the periodic cycle $\{x_0, x_1, x_2\}$ and the local degree of $\widehat{F}$ at x_k is d_k for every $k \in \{0, 1, 2\}$. Remark that the map $\psi \circ \widehat{F} : \mathbb{S}^2 \to \widehat{\mathbb{C}}$ induces a complex structure on $\mathbb{S}^2$. In other words, the uniformization theorem gives a homeomorphism $\psi' : \mathbb{S}^2 \to \widehat{\mathbb{C}}$ such that the map $\widehat{f} = \psi \circ \widehat{F} \circ \psi'^{-1}$ is holomorphic on $\widehat{\mathbb{C}}$ and thus a rational map of degree $\widehat{d}$. Moreover, up to postcomposition with a Möbius map, we may assume that $\psi'(x_k) = \widehat{z_k}$ for every $k \in \{0, 1, 2\}$. It follows that $\widehat{f}$ satisfies **(i)**.

Now remark that every connected components of the immediate super-attracting bassin of $\widehat{f}$ is simply connected since there is no critical point outside the super-attracting periodic cycle $\{\widehat{z_0}, \widehat{z_1}, \widehat{z_2}\}$. It follows that $\widehat{f}$ satisfies **(ii)**.

Finally let $\widehat{g}$ be another rational map of degree $\widehat{d}$ which satisfies **(i)** and **(ii)** for the same super-attracting periodic cycle $\{\widehat{z_0}, \widehat{z_1}, \widehat{z_2}\}$. The map $z \mapsto \widehat{f}(z) - \widehat{g}(z)$ is a rational map of degree at most $2\widehat{d}$ for which 0 has at least $d_0 + d_1 + d_2 = 2\widehat{d} + 1$ preimages counted with multiplicity (each $\widehat{z_k}$ is a preimage of 0 with multiplicity d_k). Consequently this map is identically equal to 0, that is $\widehat{f} = \widehat{g}$. □

Notice that the previous proof strongly uses the fact that the postcritical set contains only three points. Indeed if the postcritical set contains more than three points, that may be not possible to find a uniformization map ψ' for $\mathbb{S}^2$ equipped with the complex structure coming from $\psi \circ \widehat{F}$ such that $\psi'(x_k) = \widehat{z_k}$ for every $k \in \{0, 1, 2\}$. In fact we would also need to check that the topological model $\widehat{F}$ coming from Lemma 1 has no Thurston obstruction.

3.2 Cutting along a system of equipotentials

Starting with the map $\widehat{f}$ coming from Lemma 2, we need to divide $\widehat{\mathbb{C}}$ into several pieces on which the map f (or more precisely a quasiregular model F) will be piecewisely defined. This partition comes from a certain system of equipotentials of $\widehat{f}$ defined in Lemma 3.

For every $k \in \{0, 1, 2\}$, denote by $B(\widehat{z_k})$ the connected component containing $\widehat{z_k}$ of the immediate super-attracting bassin of $\widehat{f}$. Recall that each $B(\widehat{z_k})$ is simply connected. More precisely the well-known theorem of Böttcher provides Riemann mappings (namely biholomorphic maps) $\phi_k : \mathbb{D} \to B(\widehat{z_k})$ such that the following diagram commutes

$$\begin{tikzcd}
B(\widehat{z_0}) \arrow[d, "\widehat{f}"'] & \mathbb{D} \arrow[l, "\phi_0"'] \arrow[d, "z \mapsto z^{d_0}"] \\
B(\widehat{z_1}) \arrow[d, "\widehat{f}"'] & \mathbb{D} \arrow[l, "\phi_1"'] \arrow[d, "z \mapsto z^{d_1}"] \\
B(\widehat{z_2}) \arrow[d, "\widehat{f}"'] & \mathbb{D} \arrow[l, "\phi_2"'] \arrow[d, "z \mapsto z^{d_2}"] \\
B(\widehat{z_0}) & \mathbb{D} \arrow[l, "\phi_0"']
\end{tikzcd}$$

Recall that an equipotential β in any $B(\widehat{z_k})$ is the image by ϕ_k of a regular circle in $\mathbb{D}$ centered at 0. The radius of this circle is called the level of β and is denoted by $|\phi_k^{-1}(\beta)|$.

Recall that any pair of disjoint continua β, β' (that are two disjoint non-empty compact connected subsets of $\widehat{\mathbb{C}}$) uniquely defines an open annulus in $\widehat{\mathbb{C}}$ denoted by $A(\beta, \beta')$. If β, β' contain at least two points each, $A(\beta, \beta')$ is biholomorphic to a round annulus of the form $A_r = \{z \in \mathbb{C} \,/\, r < |z| < 1\}$ where $r \in]0, 1[$ only depends on $A(\beta, \beta')$. The modulus of $A(\beta, \beta')$ is defined to be $\mathrm{mod}(A(\beta, \beta')) = \frac{1}{2\pi} \log(\frac{1}{r})$. In particular if β, β' are two equipotentials in the same domain $B(\widehat{z_k})$ of levels $|\phi_k^{-1}(\beta)| > |\phi_k^{-1}(\beta')|$ then

$$\mathrm{mod}(A(\beta, \beta')) = \frac{1}{2\pi} \log \left(\frac{|\phi_k^{-1}(\beta)|}{|\phi_k^{-1}(\beta')|} \right)$$

Finally for every $k \in \{0, 1, 2\}$, denote by α_k the compact connected subset of $J(\widehat{f})$ which corresponds to the boundary of $B(\widehat{z_k})$.

Lemma 3. *If (H2) holds then there exist three equipotentials $\beta_0, \beta_1, \beta_2$ in $B(\widehat{z_0}), B(\widehat{z_1}), B(\widehat{z_2})$ respectively, together with two equipotentials β_3^+, β_3^- in $B(\widehat{z_0})$ such that*

$$|\phi_0^{-1}(\beta_0)| > |\phi_0^{-1}(\beta_3^+)| > |\phi_0^{-1}(\beta_3^-)|$$

and the following system of inequalities holds

$$\left\{ \begin{array}{rcl} \dfrac{1}{d_0} \mathrm{mod}(A(\alpha_1, \beta_1)) & < & \mathrm{mod}(A(\alpha_0, \beta_0)) \\ \dfrac{1}{d_1} \mathrm{mod}(A(\alpha_2, \beta_2)) & < & \mathrm{mod}(A(\alpha_1, \beta_1)) \\ \dfrac{1}{d_2} \mathrm{mod}(A(\alpha_0, \beta_0)) + \dfrac{1}{d_2} \mathrm{mod}(A(\beta_3^+, \beta_3^-)) & < & \mathrm{mod}(A(\alpha_2, \beta_2)) \\ \dfrac{1}{d_3} \mathrm{mod}(A(\beta_1, \beta_0)) & < & \mathrm{mod}(A(\beta_3^+, \beta_3^-)) \\ & & \\ \textit{and} \quad \mathrm{mod}(A(\beta_0, \beta_3^+)) > 1 & & \end{array} \right. \tag{2}$$

Recall that modulus is a conformal invariant, or more precisely if there is a holomorphic covering of degree d from an annulus A onto another annulus A' then $\mathrm{mod}(A) = \frac{1}{d} \mathrm{mod}(A')$. Hence the first three inequalities in (2) implies that the preimages under $\widehat{f}$ of these equipotentials are arranged as shown in Figure 5. The fourth inequality will allow to realize the preimage of the branching point α in the Hubbard tree $\mathcal{H}_P$ (see Lemma 7). The last inequality ensures sufficient space to realize the folding corresponding to the critical point c_0 in the Hubbard tree $\mathcal{H}_P$ (see Lemma 6).

The key point of the proof of Lemma 3 needs the following useful result due to Cui Guizhen and Tan Lei.

Lemma 4 (Inverse Grötzsch's inequality)**.** *Let B, B' be two disjoint topological disks in $\widehat{\mathbb{C}}$ whose boundary are respectively denoted by α, α'. Then there exists a positive constant $C > 0$ such that for every pair of equipotentials β in B and β' in B' the following inequalities hold*

$$\mathrm{mod}(A(\alpha, \beta)) + \mathrm{mod}(A(\alpha', \beta')) \leqslant \mathrm{mod}(A(\beta, \beta')) \leqslant \mathrm{mod}(A(\alpha, \beta)) + \mathrm{mod}(A(\alpha', \beta')) + C$$

The left hand side is the classical Grötzsch's inequality. The right hand side is a consequence of Koebe 1/4-theorem. We refer the readers to [CT11] for a complete proof.

Proof of Lemma 3. Let $C > 0$ be the constant coming from Lemma 4 for $B(\widehat{z_0}), B(\widehat{z_1})$. In particular, for every pair of equipotentials β_0, β_1 in $B(\widehat{z_0}), B(\widehat{z_1})$ respectively , we have

$$\frac{1}{d_3} \mathrm{mod}(A(\beta_1, \beta_0)) \leqslant \frac{1}{d_3} (\mathrm{mod}(A(\alpha_0, \beta_0)) + \mathrm{mod}(A(\alpha_1, \beta_1)) + C)$$

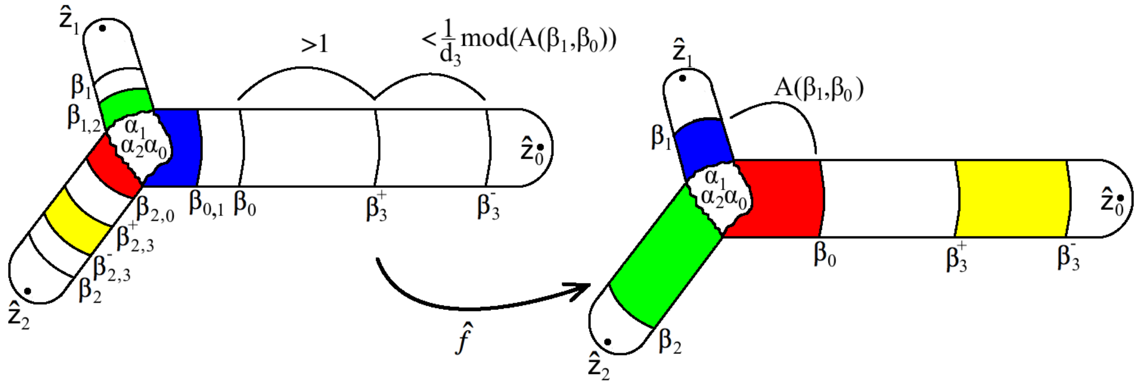


FIGURE 5: The pattern of the equipotentials (and their preimages) coming from Lemma 3 displayed on the Riemann sphere which is topologically distorted to emphasize the three domains $B(\widehat{z_0}), B(\widehat{z_1}), B(\widehat{z_2})$ (compare with Figure 4)

Now consider the following system of linear inequations with real unknows x_0, x_1, x_2, x_3.

$$\begin{cases} \frac{1}{d_0}x_1 & < & x_0 \\ \frac{1}{d_1}x_2 & < & x_1 \\ \frac{1}{d_2}x_0 + \frac{1}{d_2}x_3 & < & x_2 \\ \frac{1}{d_3}(x_0 + x_1 + C) & < & x_3 \end{cases} \tag{3}$$

Using the transition matrix M coming from Definition 1, this system is equivalent to

$$MX + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{C}{d_3} \end{pmatrix} < X \quad \text{where} \quad X = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Recall that assumption (H2) says that the leading eigenvalue $\lambda(\mathcal{H}_P, w)$ of M is less than 1. It follows from Perron-Frobenius theorem and continuity of spectral radius the existence of a vector $V \in \mathbb{R}^4$ with positive entries such that $MV < V$. Now taking $\mu > 0$ large enough (for instance $\mu = (\frac{C}{d_3} + 1)(v_3 - \frac{1}{d_3}v_0 - \frac{1}{d_3}v_1)^{-1}$), the vector $X = \mu V$ with positive entries solves the system of linear inequations (3).

The equipotentials $\beta_0, \beta_1, \beta_2$ are uniquely defined by

$$\frac{1}{2\pi}\log\left(\frac{1}{|\phi_k^{-1}(\beta_k)|}\right) = \text{mod}(A(\alpha_k, \beta_k)) = x_k \quad \text{for every } k \in \{0, 1, 2\}$$

For β_3^+, choose an arbitrary equipotential in $B(\widehat{z_0})$ such that

$$|\phi_0^{-1}(\beta_0)| > |\phi_0^{-1}(\beta_3^+)| \quad \text{and} \quad \frac{1}{2\pi}\log\left(\frac{|\phi_0^{-1}(\beta_0)|}{|\phi_0^{-1}(\beta_3^+)|}\right) = \text{mod}(A(\beta_0, \beta_3^+)) > 1$$

Then β_3^- is uniquely defined by

$$|\phi_0^{-1}(\beta_3^+)| > |\phi_0^{-1}(\beta_3^-)| \quad \text{and} \quad \frac{1}{2\pi}\log\left(\frac{|\phi_0^{-1}(\beta_3^+)|}{|\phi_0^{-1}(\beta_3^-)|}\right) = \text{mod}(A(\beta_3^+, \beta_3^-)) = x_3$$

It follows from construction that $\beta_0, \beta_1, \beta_2, \beta_3^+, \beta_3^-$ satisfy all the requirements of Lemma 3, the fourth inequality in (2) coming from the last inequality in (3) and Lemma 4. $\square$

It turns out in the proof above that the lower bound of the last inequality in (2) may be changed for any positive constant (which depends only on the integers d_0, d_1, d_2, d_3). As we will see later in Lemma 5 and Lemma 6, the lower bound 1 ensures sufficient space to make the surgery in $A(\beta_0, \beta_3^+)$. However, the author guesses that the last inequality in (2) is not necessary (see discussion after the proof of Lemma 5).

The system of equipotentials coming from Lemma 3 will be used to divide $\widehat{\mathbb{C}}$ into several pieces on which a quasiregular map F will be piecewisely defined. This map F should be carefully defined in such a way that its dynamics is encoded by the weighted Hubbard tree $(\mathcal{H}_P, w)$ (see Theorem 3).

For instance, the first step of the construction which corresponds to the dynamics on $e_1 \cup e_2$ for $\mathcal{H}_P$ (see Figure 4) is the following. Denote by $\beta_{0,1}$ the preimage of β_1 in $B(\widehat{z_0})$ (see Figure 5). From Lemma 3, $\beta_{0,1}$ is an equipotential of level $|\phi_0^{-1}(\beta_{0,1})| > |\phi_0^{-1}(\beta_0)|$. Denote by $D(\beta_{0,1})$ the open disk bounded by $\beta_{0,1}$ and containing $\{\widehat{z_1}, \widehat{z_2}\}$ (and hence $J(\widehat{f}) \cup B(\widehat{z_1}) \cup B(\widehat{z_2})$ as well). Then F is defined to be the rational map $\widehat{f}$ on $D(\beta_{0,1})$. Remark that $F|_{D(\beta_{0,1})}$ continuously extends to $\beta_{0,1}$ by a degree d_0 covering denoted by $F|_{\beta_{0,1}} : \beta_{0,1} \to \beta_1$.

3.3 Folding with an annulus-disk surgery

The aim of this part of the construction is to realize the folding corresponding to the critical point c_0 in the Hubbard tree $\mathcal{H}_P$ (see Figure 4). More precisely F should holomorphically maps a small annulus (corresponding to a neighborhood of c_0 in $\mathcal{H}_P$) onto a disk (corresponding to a neighborhood of c_1 in $\mathcal{H}_P$) with respect to the degrees d_0, d_3. Lemma 5 below provides a general result about the existence of such a map. Then Lemma 6 is a direct application of Lemma 5 for our construction.

Lemma 5 (Annulus-disk map)**.** *Let n, n' be two positive integers. Then there exists a holomorphic branched covering $G : A(\gamma, \gamma') \to \mathbb{D}$ from an annulus bounded by two quasicircles γ, γ' onto the open unit disk $\mathbb{D}$ centered at 0 and of radius 1 such that*

(i) *G is of degree $n + n'$ and has $n + n'$ critical points counted with multiplicity*

(ii) *G continuously extends to $\gamma \cup \gamma'$ by a degree n covering $G|_\gamma : \gamma \to \partial\mathbb{D}$ and a degree n' covering $G|_{\gamma'} : \gamma' \to \partial\mathbb{D}$*

(iii) $\mathrm{mod}(A(\gamma, \gamma')) \leqslant 1$

Proof. There are many ways to prove the existence of such a map. Here this proof uses the properties of the McMullen's family

$$g_{0,\lambda} : z \mapsto z^n + \frac{\lambda}{z^{n'}}$$

for $|\lambda| > 0$ small enough (see [McM88] and [DHL+08] for a complete study of this family). Recall that $g_{0,\lambda}$ is of degree $n + n'$, its critical set contains $n + n'$ simple critical points of the form

$$c_k = \left(\frac{n'}{n}\right)^{1/(n+n')} |\lambda|^{1/(n+n')} e^{2ki\pi/(n+n')} \quad \text{where } k \in \{1, 2, \dots, n + n'\}$$

(the other critical points are ∞ of multiplicity $n-1$ if $n>1$ and 0 of multiplicity $n'-1$ if $n'>1$). Moreover, the preimages of 0 are of the form

$$g_{0,\lambda}^{-1}(0) = \{|\lambda|^{1/(n+n')} e^{2ki\pi/(n+n')} \,/\, k \in \{1, 2, \ldots, n+n'\}\}$$

Let A be the preimage of the open unit disk $\mathbb{D}$, namely $A = g_{0,\lambda}^{-1}(\mathbb{D})$. We are going to prove that for every $|\lambda| > 0$ small enough A is connected and actually an annulus separating 0 and ∞. Indeed remark that for every $z \in \mathbb{C}$ with modulus $|z| = |\lambda|^{1/(n+n')}$ we have

$$|g_{0,\lambda}(z)| = \left| z^n + \frac{\lambda}{z^{n'}} \right| \leqslant |\lambda|^{n/(n+n')} + \frac{|\lambda|}{|\lambda|^{n'/(n+n')}} = 2|\lambda|^{n/(n+n')}$$

Similarly for every $k \in \{1, 2, \ldots, n+n'\}$ we have

$$|g_{0,\lambda}(c_k)| \leqslant \left(\frac{n'}{n}\right)^{n/(n+n')} |\lambda|^{n/(n+n')} + \left(\frac{n}{n'}\right)^{n'/(n+n')} \frac{|\lambda|}{|\lambda|^{n'/(n+n')}} = C|\lambda|^{n/(n+n')}$$

with

$$C = \left(\frac{n'}{n}\right)^{n/(n+n')} + \left(\frac{n}{n'}\right)^{n'/(n+n')} = \left(\frac{n+n'}{n}\right)^{n/(n+n')} \times \left(\frac{n+n'}{n'}\right)^{n'/(n+n')} \leqslant 2$$

(by using the arithmetic-geometric mean inequality $x^{n/(n+n')} \times y^{n'/(n+n')} \leqslant \frac{n}{n+n'}x + \frac{n'}{n+n'}y$).

So if λ is such that $0 < 2|\lambda|^{n/(n+n')} < 1$ then A contains the circle centered at 0 and of radius $|\lambda|^{1/(n+n')}$ where all the preimages of 0 lie, together with $n+n'$ simple critical points of $g_{0,\lambda}$. In particular, A is a connected set which separates 0 and ∞ and it follows from the Riemann-Hurwitz formula applied to the degree $n+n'$ branched covering $g_{0,\lambda}|_A : A \to \mathbb{D}$ that A is an annulus.

Let γ be the outer boundary of A, namely the boundary of the connected component of $\widehat{\mathbb{C}} - \bar{A}$ containing ∞, and γ' be the inner boundary of A, namely the boundary of the connected component of $\widehat{\mathbb{C}} - \bar{A}$ containing 0. It turns out that $A = A(\gamma, \gamma')$ and $G = g_{0,\lambda}|_A : A(\gamma, \gamma') \to \mathbb{D}$ satisfy **(i)**. The point **(ii)** follows from the fact that $g_{0,\lambda}$ realizes a degree n (respectively n') branched covering on the the connected component of $\widehat{\mathbb{C}} - \bar{A}$ containing ∞ (respectively 0) with no critical points on the boundary. Moreover γ and γ' are quasicircles as preimages of the unit circle $\partial\mathbb{D}$ by conformal maps.

For the point **(iii)**, remark that for every $R \geqslant 1$ we have

$$|z| \leqslant \frac{1}{R^{1/n'}}|\lambda|^{1/(n+n')} \Rightarrow |g_{0,\lambda}(z)| \geqslant \frac{|\lambda|}{|z|^{n'}} - |z|^n \geqslant |\lambda|^{n/(n+n')}\left(R - \frac{1}{R^n}\right)$$

$$|z| \geqslant R^{1/n}|\lambda|^{1/(n+n')} \Rightarrow |g_{0,\lambda}(z)| \geqslant |z|^n - \frac{|\lambda|}{|z|^{n'}} \geqslant |\lambda|^{n/(n+n')}\left(R - \frac{1}{R^{n'}}\right)$$

In particular if $R = 2|\lambda|^{-n/(n+n')}$, then $\max\{\frac{1}{R^n}, \frac{1}{R^{n'}}\} \leqslant \frac{1}{R} < \frac{1}{2}R$ (since λ was chosen so that $0 < 2|\lambda|^{n/(n+n')} < 1$ that implies $R > 4$) and hence

$$|g_{0,\lambda}(z)| > |\lambda|^{n/(n+n')}\frac{1}{2}R = 1 \quad \text{if } |z| \leqslant \frac{1}{R^{1/n'}}|\lambda|^{1/(n+n')} \text{ or } |z| \geqslant R^{1/n}|\lambda|^{1/(n+n')}$$

Consequently the preimage $A = A(\gamma, \gamma')$ of the unit disk is contained as essential subannulus in the round annulus $\{z \in \mathbb{C} \,/\, \frac{1}{R^{1/n'}}|\lambda|^{1/(n+n')} < |z| < R^{1/n}|\lambda|^{1/(n+n')}\}$ and the Grötzsch's inequality gives

$$\operatorname{mod}(A(\gamma, \gamma')) \leqslant \frac{1}{2\pi}\log\left(\frac{R^{1/n}|\lambda|^{1/(n+n')}}{\frac{1}{R^{1/n'}}|\lambda|^{1/(n+n')}}\right) = \frac{1}{2\pi}\left(\frac{1}{n} + \frac{1}{n'}\right)\log(R)$$

In particular, if λ is fixed so that $2|\lambda|^{n/(n+n')} = \frac{4}{e^\pi} < 1$ then $R = 2|\lambda|^{-n/(n+n')} = e^\pi$ and

$$\operatorname{mod}(A(\gamma,\gamma')) \leqslant \frac{1}{2}\left(\frac{1}{n}+\frac{1}{n'}\right) \leqslant 1$$

□

In the proof above, 1 is obviously not the optimal upper bound for $\operatorname{mod}(A(\gamma,\gamma'))$. The author guesses that this modulus is arbitrary small when λ is close to 0. But one can prove that the modulus of the smallest round annulus (namely an annulus bounded by two disjoint euclidean circles centered at 0) containing $A(\gamma,\gamma')$ as essential subannulus is bounded by below by a positive constant which does not depend on λ. The same happens if the open unit disk $\mathbb{D}$ is replaced by any open disk centered at 0 and containing the critical values. However we do not need a sharper estimation than **(iii)** in this paper (see Lemma 3 and the proof of Lemma 6 below).

Now we will apply Lemma 5 to realize the folding corresponding to the critical point c_0 in the Hubbard tree $\mathcal{H}_P$.

Let γ_1 be an arbitrary equipotential in $B(\widehat{z_1})$ such that $|\phi_1^{-1}(\gamma_1)| < |\phi_1^{-1}(\beta_1)|$. In order to follow more easily the construction, we will slightly improve the notation. So let $\gamma_{0,1}$ be the equipotential β_0, keeping in mind that $\gamma_{0,1}$ will be mapped onto γ_1 by a degree d_0 covering. Notice that the first inequality of (2) in Lemma 3 implies $|\phi_0^{-1}(\beta_{0,1})| > |\phi_0^{-1}(\gamma_{0,1})|$. Similarly let $\beta_{3,1}$ be the equipotential β_3^+, keeping in mind that $\beta_{3,1}$ will be mapped onto β_1 by a degree d_3 covering.

Lemma 6. *There exist an equipotential $\gamma_{3,1}$ in $B(\widehat{z_0})$ and a holomorphic branched covering $F|_{A(\gamma_{0,1},\gamma_{3,1})} : A(\gamma_{0,1},\gamma_{3,1}) \to D(\gamma_1)$ such that*

(i) $|\phi_0^{-1}(\beta_{0,1})| > |\phi_0^{-1}(\gamma_{0,1})| > |\phi_0^{-1}(\gamma_{3,1})| > |\phi_0^{-1}(\beta_{3,1})|$

(ii) *the image of $A(\gamma_{0,1},\gamma_{3,1})$ by $F|_{A(\gamma_{0,1},\gamma_{3,1})}$ is the open disk $D(\gamma_1)$ bounded by γ_1 and containing $\widehat{z_1}$*

(iii) *$F|_{A(\gamma_{0,1},\gamma_{3,1})}$ is of degree d_0+d_3 and has d_0+d_3 critical points counted with multiplicity, furthermore one of them, denoted by c, satisfies $F|_{A(\gamma_{0,1},\gamma_{3,1})}(c) = \widehat{z_1}$*

(iv) *$F|_{A(\gamma_{0,1},\gamma_{3,1})}$ continuously extends to $\gamma_{0,1}\cup\gamma_{3,1}$ by a degree d_0 covering $F|_{\gamma_{0,1}} : \gamma_{0,1} \to \gamma_1$ and a degree d_3 covering $F|_{\gamma_{3,1}} : \gamma_{3,1} \to \gamma_1$*

Proof. Let $G : A(\gamma,\gamma') \to D$ be a holomorphic branched covering coming from Lemma 5 for the integers $n = d_0$ and $n' = d_3$. Define the equipotential $\gamma_{3,1}$ by

$$|\phi_0^{-1}(\gamma_{0,1})| > |\phi_0^{-1}(\gamma_{3,1})| \quad \text{and} \quad \frac{1}{2\pi}\log\left(\frac{|\phi_0^{-1}(\gamma_{0,1})|}{|\phi_0^{-1}(\gamma_{3,1})|}\right) = \operatorname{mod}(A(\gamma_{0,1},\gamma_{3,1})) = \operatorname{mod}(A(\gamma,\gamma'))$$

Since $\operatorname{mod}(A(\gamma_{0,1},\beta_{3,1})) = \operatorname{mod}(A(\beta_0,\beta_3^+)) > 1$ (from the last inequality of (2) in Lemma 3) and $\operatorname{mod}(A(\gamma,\gamma')) \leqslant 1$ (from the point **(iii)** in Lemma 5), it follows that $|\phi_0^{-1}(\gamma_{3,1})| > |\phi_0^{-1}(\beta_{3,1})|$ and the point **(i)** holds.

Now let φ be any biholomorphic map from $A(\gamma_{0,1},\gamma_{3,1})$ onto $A(\gamma,\gamma')$ (the existence of such a biholomorphic map is ensured by the fact that these two annuli have same modulus). Since $A(\gamma_{0,1},\gamma_{3,1})$ and $A(\gamma,\gamma')$ are bounded by quasicircles, φ may be continuously extended to $\gamma_{0,1}\cup\gamma_{3,1}$ by two homeomorphisms.

Let c be the preimage under φ of any critical point of G and let $\phi : \mathbb{D} \to D(\gamma_1)$ be any Riemann mapping of $D(\gamma_1)$ such that $\phi(G(\varphi(c))) = \widehat{z_1}$. Since $D(\gamma_1)$ is bounded by a quasicircle, ϕ may be continuously extended to $\partial\mathbb{D}$ by a homeomorphism. Then $F|_{A(\gamma_{0,1},\gamma_{3,1})} = \phi \circ G \circ \varphi$ is holomorphic on $A(\gamma_{0,1}, \gamma_{3,1})$ and satisfies **(ii)**, **(iii)**, and **(iv)** by construction. $\square$

Figure 6 depicts the equipotentials involved in Lemma 6.

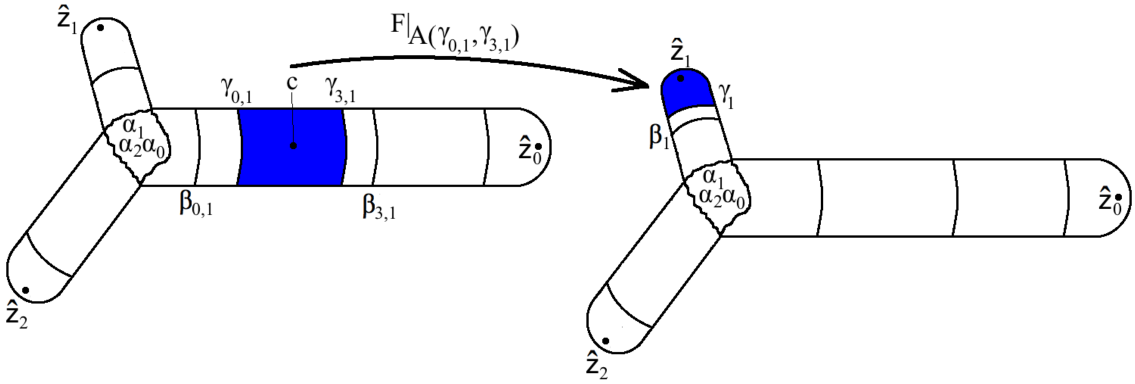


FIGURE 6: The map $F|_{A(\gamma_{0,1},\gamma_{3,1})}$ coming from Lemma 6 displayed on the Riemann sphere which is topologically distorted to emphasize the three domains $B(\widehat{z_0}), B(\widehat{z_1}), B(\widehat{z_2})$ (compare with Figure 4)

3.4 Preimage of the the branching part

According to the last two sections, the map F is defined up to there on the union of the open disk $D(\beta_{0,1})$ containing $\{\widehat{z_1}, \widehat{z_2}\}$ with the open annulus $A(\gamma_{0,1}, \gamma_{3,1})$ containing c. Moreover F maps c to $\widehat{z_1}$, $\widehat{z_1}$ to $\widehat{z_2}$ and $\widehat{z_2}$ to $\widehat{z_0}$. Now we need to define F near $\widehat{z_0}$ by sending $\widehat{z_0}$ to c in order to realize a cycle of period 4 as required in Theorem 3. This should be done carefully so that the quasiconformal surgery may be concluded.

The first problem is that some preimage of $J(\widehat{f})$ (or more precisely of the annulus $A(\beta_1, \beta_0)$ containing $J(\widehat{f})$) must appear in $B(\widehat{z_0})$ (compare with Figure 4 where the edge $e_3 = [c_0, c_3]_{\mathcal{H}_P}$ contains a preimage of the branching point α). This is done in Lemma 7 below which essentially uses the fourth inequality of (2) in Lemma 3.

Lemma 7. *There exist an equipotential $\beta_{3,0}$ in $B(\widehat{z_0})$ and a holomorphic covering $F|_{A(\beta_{3,1},\beta_{3,0})} : A(\beta_{3,1}, \beta_{3,0}) \to A(\beta_1, \beta_0)$ such that*

(i) $|\phi_0^{-1}|(\beta_{3,1})| = |\phi_0^{-1}|(\beta_3^+)| > |\phi_0^{-1}|(\beta_{3,0})| > |\phi_0^{-1}|(\beta_3^-)|$

(ii) *the image of $A(\beta_{3,1}, \beta_{3,0})$ by $F|_{A(\beta_{3,1},\beta_{3,0})}$ is the open annulus $A(\beta_1, \beta_0)$*

(iii) *$F|_{A(\beta_{3,1},\beta_{3,0})}$ is of degree d_3 and has no critical point*

(iv) *$F|_{A(\beta_{3,1},\beta_{3,0})}$ continuously extends to $\beta_{3,1} \cup \beta_{3,0}$ by two degree d_3 coverings $F|_{\beta_{3,1}} : \beta_{3,1} \to \beta_1$ and $F|_{\beta_{3,0}} : \beta_{3,0} \to \beta_0$*

Proof. Define the equipotential $\beta_{3,0}$ by

$$|\phi_0^{-1}(\beta_{3,1})| > |\phi_0^{-1}(\beta_{3,0})| \quad \text{and} \quad \frac{1}{2\pi} \log\left(\frac{|\phi_0^{-1}(\beta_{3,1})|}{|\phi_0^{-1}(\beta_{3,0})|}\right) = \operatorname{mod}(A(\beta_{3,1}, \beta_{3,0})) = \frac{1}{d_3} \operatorname{mod}(A(\beta_1, \beta_0))$$

Since $\beta_{3,1} = \beta_3^+$ (see the previous subsection) and $\frac{1}{d_3}\,\mathrm{mod}(A(\beta_1, \beta_0)) < \mathrm{mod}(A(\beta_3^+, \beta_3^-))$ (from the fourth inequality of (2) in Lemma 3), it follows that $|\phi_0^{-1}|(\beta_{3,0})| > |\phi_0^{-1}|(\beta_3^-)|$ and the point **(i)** holds.

Now let φ be any biholomorphic map from $A(\beta_{3,1}, \beta_{3,0})$ onto a round annulus of the form $A_r = \{z \in \mathbb{C} \,/\, r < |z| < 1\}$ where r is defined by

$$\frac{1}{2\pi} \log\left(\frac{1}{r}\right) = \mathrm{mod}(A(\beta_{3,1}, \beta_{3,0}))$$

Since $A(\beta_{3,1}, \beta_{3,0})$ is bounded by equipotentials, φ may be continuously extended to $\beta_{3,1} \cup \beta_{3,0}$ by two homeomorphisms which send $\beta_{3,1}$ onto $\{z \in \mathbb{C} \,/\, |z| = 1\}$ and $\beta_{3,0}$ onto $\{z \in \mathbb{C} \,/\, |z| = r\}$. Similarly, let ψ be any biholomorphic map from $A(\beta_1, \beta_0)$ onto $A_{r^{d_3}}$. The existence of such a biholomorphic map is ensured by the following remark

$$\mathrm{mod}(A(\beta_1, \beta_0)) = d_3\,\mathrm{mod}(A(\beta_{3,1}, \beta_{3,0})) = \frac{d_3}{2\pi} \log\left(\frac{1}{r}\right) = \frac{1}{2\pi} \log\left(\frac{1}{r^{d_3}}\right)$$

ψ may be continuously extended to $\beta_1 \cup \beta_0$ by two homeomorphisms which send β_1 onto $\{z \in \mathbb{C} \,/\, |z| = 1\}$ and β_0 onto $\{z \in \mathbb{C} \,/\, |z| = r^{d_3}\}$.

Then $F|_{A(\beta_{3,1},\beta_{3,0})} = \psi^{-1} \circ (z \mapsto z^{d_3}) \circ \varphi$ is holomorphic on $A(\beta_{3,1}, \beta_{3,0})$ and satisfies **(ii)**, **(iii)** and **(iv)** by construction. $\square$

Figure 7 depicts the map $F|_{A(\beta_{3,1},\beta_{3,0})}$ coming from Lemma 7.

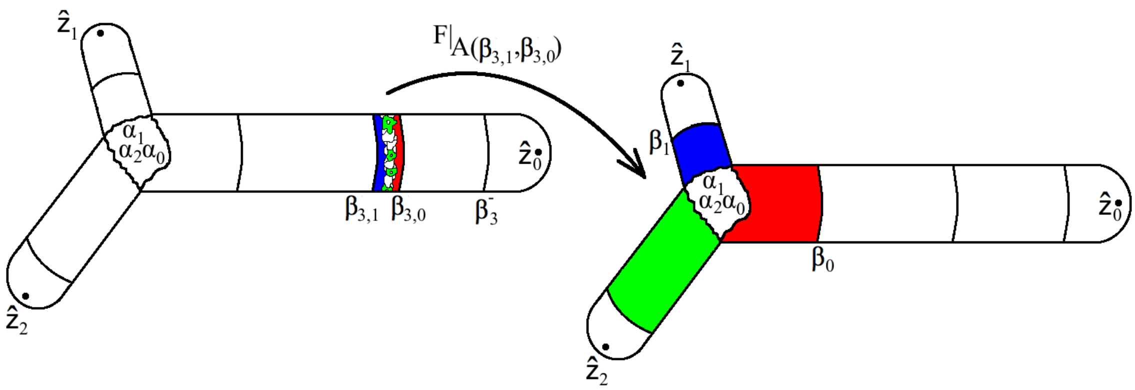


FIGURE 7: The map $F|_{A(\beta_{3,1},\beta_{3,0})}$ coming from Lemma 7 displayed on the Riemann sphere which is topologically distorted to emphasize the three domains $B(\widehat{z_0}), B(\widehat{z_1}), B(\widehat{z_2})$ (compare with Figure 4)

3.5 Achievement of the super-attracting cycle of period 4

Now we achieve the definition of F near $\widehat{z_0}$. This is done in two parts. Firstly Lemma 9 realizes a preimage of a neighborhood of $\widehat{z_0}$ which does not contain the critical point c (coming from Lemma 6). Then Lemma 10 defines F near $\widehat{z_0}$ by sending a neighborhood of $\widehat{z_0}$ onto a neighborhood of c (mapping $\widehat{z_0}$ to c). Before, Lemma 8 is a technical result from conformal geometry which allows to define precisely the neighborhoods involved in this construction.

Lemma 8 (Separating quasicircle)**.** *Let $A(\gamma, \gamma')$ be an open annulus bounded by a pair of disjoint quasicircles γ, γ', and let a be a point in $A(\gamma, \gamma')$. Then there exists a quasicircle δ in $A(\gamma, \gamma')$ which separates a from $\gamma \cup \gamma'$ such that* $\mathrm{mod}(A(\gamma, \delta))$ *is arbitrary small.*

Proof. This proofs uses the definition of the modulus of an annulus by the extremal length (see [Ahl10]). Up to a biholomorphic change of coordinates, we may assume that $\gamma = \{z \in \mathbb{C} \,/\, |z| = 1\}$, $\gamma' = \{z \in \mathbb{C} \,/\, |z| = e^{-2\pi \operatorname{mod}(A(\gamma,\gamma'))} < 1\}$ and a is a positive real number in $]e^{-2\pi \operatorname{mod}(A(\gamma,\gamma'))}, 1[$. Fix x to be the positive real number $x = (1 + e^{-2\pi \operatorname{mod}(A(\gamma,\gamma'))})/2$ and define δ_ε to be the euclidean circle centered at x and of radius $1 - x - \varepsilon$ for every $\varepsilon > 0$. Notice that δ_ε is included in $A(\gamma, \gamma')$ and that δ_ε separates a from $\gamma \cup \gamma'$ for every $\varepsilon > 0$ small enough.

For every angle θ (small enough), consider the path ℓ_θ connecting δ_ε and γ of the form $\ell_\theta = \{z = re^{i\theta} \,/\, R_\theta \leqslant r \leqslant 1\}$ with $R_\theta > 0$ maximal such that $R_\theta e^{i\theta} \in \delta_\varepsilon$. By classical results from euclidean geometry and trigonometry, we get:

$$R_\theta = x\cos(\theta) + \sqrt{(1 - x - \varepsilon)^2 - x^2 \sin^2(\theta)}$$

Since $\theta \mapsto R_\theta$ is an even function with a local maximum at $\theta = 0$, it follows for every $\varepsilon > 0$ small enough that

$$\begin{aligned} \theta \in [-\sqrt{\varepsilon}, \sqrt{\varepsilon}] \Longrightarrow R_\theta \geqslant R_{\sqrt{\varepsilon}} &= x\cos(\sqrt{\varepsilon}) + \sqrt{(1 - x - \varepsilon)^2 - x^2 \sin^2(\sqrt{\varepsilon})} \\ &= 1 - \frac{2 - x}{2(1 - x)}\varepsilon + \underset{\varepsilon \to 0}{O}(\varepsilon^2) \\ &\geqslant 1 - C\varepsilon \qquad (4) \end{aligned}$$

where C is a positive constant fixed so that $C > \frac{2-x}{2(1-x)}$.

Now recall that the modulus of $A(\gamma, \delta_\varepsilon)$ is given by the extremal length of the collection L of rectifiable paths connecting δ_ε and γ, namely

$$\operatorname{mod}(A(\gamma, \delta_\varepsilon)) = \sup_\rho \frac{\left(\inf_{\ell \in L} \int_\ell \rho |dz|\right)^2}{\int_{A(\gamma,\delta_\varepsilon)} \rho^2 dxdy}$$

where the supremum is over all measurable functions $\rho : A(\gamma, \delta_\varepsilon) \to [0, +\infty]$ such that $\int_{A(\gamma,\delta_\varepsilon)} \rho^2 dxdy < +\infty$. Let ρ be such a measurable function. For every θ (small enough), we have:

$$\inf_{\ell \in L} \int_\ell \rho |dz| \leqslant \int_{\ell_\theta} \rho |dz| = \int_{R_\theta}^1 \rho(re^{i\theta}) dr$$

Integrating over $\theta \in [-\sqrt{\varepsilon}, \sqrt{\varepsilon}]$ and applying the Cauchy-Schwarz inequality give

$$\begin{aligned} 2\sqrt{\varepsilon} \inf_{\ell \in L} \int_\ell \rho |dz| &\leqslant \left(\int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \int_{R_\theta}^1 \rho(re^{i\theta})^2 r dr d\theta \right)^{1/2} \left(\int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \int_{R_\theta}^1 \frac{1}{r} dr d\theta \right)^{1/2} \\ &\leqslant \left(\int_{A(\gamma,\delta_\varepsilon)} \rho^2 dxdy \right)^{1/2} \left(\int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \log\left(\frac{1}{R_\theta}\right) d\theta \right)^{1/2} \end{aligned}$$

Therefore it follows from (4) that

$$\begin{aligned} \frac{\left(\inf_{\ell \in L} \int_\ell \rho |dz|\right)^2}{\int_{A(\gamma,\delta_\varepsilon)} \rho^2 dxdy} &\leqslant \frac{1}{4\varepsilon} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \log\left(\frac{1}{R_\theta}\right) d\theta \\ &\leqslant \frac{1}{4\varepsilon} \int_{-\sqrt{\varepsilon}}^{\sqrt{\varepsilon}} \log\left(\frac{1}{1 - C\varepsilon}\right) d\theta = \frac{1}{2\sqrt{\varepsilon}} \log\left(\frac{1}{1 - C\varepsilon}\right) \end{aligned}$$

Finally we take the supremum over all measurable functions ρ to get

$$\operatorname{mod}(A(\gamma,\delta_\varepsilon)) \leqslant \frac{1}{2\sqrt{\varepsilon}} \log\left(\frac{1}{1-C\varepsilon}\right) \underset{\varepsilon\to 0}{\sim} \frac{C}{2}\sqrt{\varepsilon} \underset{\varepsilon\to 0}{\to} 0$$

that concludes the proof. □

Now we come back to the construction. Let γ_0 be an arbitrary equipotential in $B(\widehat{z_0})$ such that $|\phi^{-1}(\beta_0)| = |\phi_0^{-1}(\gamma_{0,1})| > |\phi_0^{-1}(\gamma_0)| > |\phi_0^{-1}(\gamma_{3,1})|$ and $A(\gamma_0,\gamma_{3,1})$ contains the critical point c.

Lemma 9. *There exist two equipotentials $\gamma_{3,0}$ and $\delta_{3,c}^+$ in $B(\widehat{z_0})$ and a holomorphic covering $F|_{A(\gamma_{3,0},\delta_{3,c}^+)} : A(\gamma_{3,0},\delta_{3,c}^+) \to A(\gamma_0,\delta_c^+)$ such that*

(i) $|\phi_0^{-1}(\beta_{3,0})| > |\phi_0^{-1}(\gamma_{3,0})| > |\phi_0^{-1}(\delta_{3,c}^+)| > |\phi_0^{-1}(\beta_3^-)|$

(ii) *the image of $A(\gamma_{3,0},\delta_{3,c}^+)$ by $F|_{A(\gamma_{3,0},\delta_{3,c}^+)}$ is an open annulus $A(\gamma_0,\delta_c^+)$ where δ_c^+ is a quasicircle in $A(\gamma_0,\gamma_{3,1})$ which separates c from $\gamma_0\cup\gamma_{3,1}$*

(iii) *$F|_{A(\gamma_{3,0},\delta_{3,c}^+)}$ is of degree d_3 and has no critical point*

(iv) *$F|_{A(\gamma_{3,0},\delta_{3,c}^+)}$ continuously extends to $\gamma_{3,0}\cup\delta_{3,c}^+$ by two degree d_3 coverings $F|_{\gamma_{3,0}} : \gamma_{3,0}\to\gamma_0$ and $F|_{\delta_{3,c}^+} : \delta_{3,c}^+ \to \delta_c^+$*

Proof. Applying Lemma 8, we get a quasicircle δ_c^+ in $A(\gamma_0,\gamma_{3,1})$ which separates c from $\gamma_0\cup\gamma_{3,1}$ and such that

$$\frac{1}{d_3}\operatorname{mod}(A(\gamma_0,\delta_c^+)) < \operatorname{mod}(A(\beta_{3,0},\beta_3^-))$$

Therefore we can find two equipotentials $\gamma_{3,0}$ and $\delta_{3,c}^+$ in $B(\widehat{z_0})$ so that

$$|\phi_0^{-1}(\beta_{3,0})| > |\phi_0^{-1}(\gamma_{3,0})| > |\phi_0^{-1}(\delta_{3,c}^+)| > |\phi_0^{-1}(\beta_3^-)|$$

$$\text{and}\quad \frac{1}{2\pi}\log\left(\frac{|\phi_0^{-1}(\gamma_{3,0})|}{|\phi_0^{-1}(\delta_{3,c}^+)|}\right) = \operatorname{mod}(A(\gamma_{3,0},\delta_{3,c}^+)) = \frac{1}{d_3}\operatorname{mod}(A(\gamma_0,\delta_c^+))$$

The point **(i)** holds by definition. For the three other points, the proof may be achieved as that one of Lemma 7. □

Figure 8 depicts the equipotentials involved in Lemma 9.

It remains to define F near $\widehat{z_0}$. Let δ_c^- be an arbitrary quasicircle in the open disk $D(\delta_c^+)$ bounded by δ_c^+ and containing c which separates c from δ_c^+. We slightly improve the notation by denoting $\delta_{3,c}^-$ the equipotential β_3^- keeping in mind that $\delta_{3,c}^-$ will be mapped onto δ_c^- by a degree d_3 covering.

Lemma 10. *There exists a degree d_3 holomorphic branched covering $F|_{D(\delta_{3,c}^-)} : D(\delta_{3,c}^-) \to D(\delta_c^-)$ from the open disk $D(\delta_{3,c}^-)$ bounded by $\delta_{3,c}^-$ and containing $\widehat{z_0}$ onto the open disk $D(\delta_c^-)$ bounded by δ_c^- and containing c such that $\widehat{z_0}$ is the only one critical point of $F|_{D(\delta_{3,c}^-)}$ with $F|_{D(\delta_{3,c}^-)}(\widehat{z_0}) = c$ and local degree d_3. Moreover $F|_{D(\delta_{3,c}^-)}$ continuously extends to $\delta_{3,c}^-$ by a degree d_3 covering denoted by $F|_{\delta_{3,c}^-} : \delta_{3,c}^- \to \delta_c^-$.*

Proof. Let $\phi : \mathbb{D} \to D(\delta_{3,c}^-)$ be any Riemann mapping of the open disk $D(\delta_{3,c}^-)$ such that $\phi(0) = \widehat{z}_0$, and let $\phi' : \mathbb{D} \to D(\delta_c^-)$ be any Riemann mapping of the open disk $D(\delta_c^-)$ such that $\phi'(0) = c$. Since $D(\delta_{3,c}^-)$ (respectively $D(\delta_c^-)$) is bounded by a quasicircle, ϕ (respectively ϕ') may be continuously extended to $\partial\mathbb{D}$ by a homeomorphism. Then $F|_{D(\delta_{3,c}^-)} = \phi' \circ (z \mapsto z^{d_3}) \circ \phi^{-1}$ gives the result. □

Figure 8 depicts the map $F|_{D(\delta_{3,c}^-)}$ coming from Lemma 10.

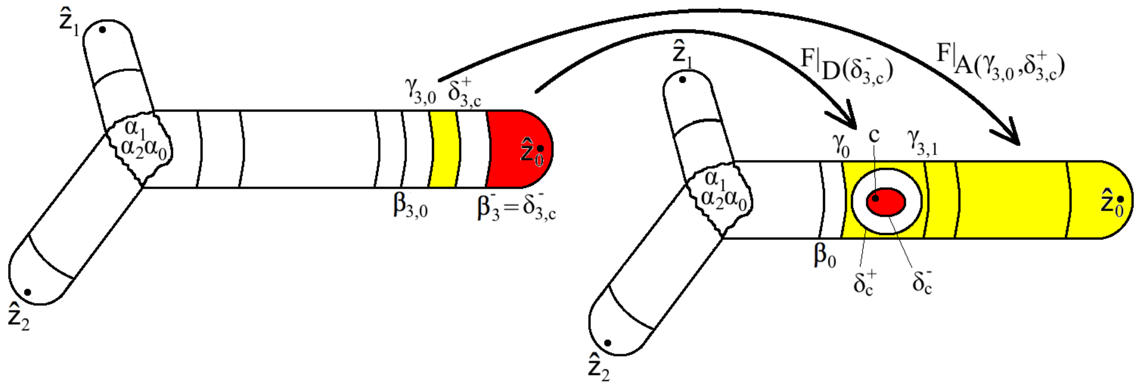


FIGURE 8: The maps $F|_{A(\gamma_{3,0},\delta_{3,c}^+)}$ and $F|_{D(\delta_{3,c}^-)}$ coming from Lemma 9 and Lemma 10 displayed on the Riemann sphere which is topologically distorted to emphasize the three domains $B(\widehat{z}_0), B(\widehat{z}_1), B(\widehat{z}_2)$ (compare with Figure 4)

3.6 Uniformization

At first we sum up in the following table the definition of F up to there.

domain of definition	image	continuous extension on the boundary	critical points	critical values
$D(\beta_{0,1})$	$\widehat{\mathbb{C}}$	$\beta_{0,1} \xrightarrow{d_0:1} \beta_1$	$\widehat{z}_1$ with mult. $d_1 - 1$ $\widehat{z}_2$ with mult. $d_2 - 1$	$F(\widehat{z}_1) = \widehat{z}_2$ $F(\widehat{z}_2) = \widehat{z}_0$
$A(\gamma_{0,1}, \gamma_{3,1})$	$D(\gamma_1)$	$\gamma_{0,1} \xrightarrow{d_0:1} \gamma_1$ $\gamma_{3,1} \xrightarrow{d_3:1} \gamma_1$	$c \in \{d_0 + d_3$ crit. pts counted with mult.$\}$	$F(c) = \widehat{z}_1$ and others
$A(\beta_{3,1}, \beta_{3,0})$	$A(\beta_1, \beta_0)$	$\beta_{3,1} \xrightarrow{d_3:1} \gamma_1$ $\beta_{3,0} \xrightarrow{d_3:1} \beta_0$	$\emptyset$	$\emptyset$
$A(\gamma_{3,0}, \delta_{3,c}^+)$	$A(\gamma_0, \delta_c^+)$	$\gamma_{3,0} \xrightarrow{d_3:1} \gamma_0$ $\delta_{3,c}^+ \xrightarrow{d_3:1} \delta_c^+$	$\emptyset$	$\emptyset$
$D(\delta_{3,c}^-)$	$D(\delta_c^-)$	$\delta_{3,c}^- \xrightarrow{d_3:1} \delta_c^-$	$\widehat{z}_0$ with mult. $d_3 - 1$	$F(\widehat{z}_0) = c$

So F is holomorphically defined on $H = D(\beta_{0,1}) \cup A(\gamma_{0,1}, \gamma_{3,1}) \cup A(\beta_{3,1}, \beta_{3,0}) \cup A(\gamma_{3,0}, \delta^+_{3,c}) \cup D(\delta^-_{3,c})$ with continuous extension on the boundary. It remains to define F on the complement $Q = \widehat{\mathbb{C}} - \overline{H} = A(\beta_{0,1}, \gamma_{0,1}) \cup A(\gamma_{3,1}, \beta_{3,1}) \cup A(\beta_{3,0}, \gamma_{3,0}) \cup A(\delta^+_{3,c}, \delta^-_{3,c})$. This is done in the following lemma.

Lemma 11. *The map $F|_{\overline{H}} : \overline{H} \to \widehat{\mathbb{C}}$ extend to a quasiregular map $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by quasiconformal coverings defined on each connected component of $Q = \widehat{\mathbb{C}} - \overline{H}$.*

Moreover there exists an open subset $E \subset H$ such that $F(E) \subset E$ and $F^2(\overline{Q}) \subset E$.

In particular, notice that the quasiregular map $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ has no more critical points than those coming from the holomorphic restriction $F|_H : H \to \widehat{\mathbb{C}}$.

Proof. Remark that every connected component of Q is an open annulus whose two connected components of the boundary are quasicircles where F realizes two coverings of same degree. Interpolating these two coverings of same degree, F may be continuously extended to each connected component of Q by a covering of degree corresponding to that one on the boundary. Since the connected components of the boundary of each connected component of Q, together with their images under F, are quasicircles, each interpolation may be carefully done in such a way that the resulting covering is actually quasiconformal. Finally F quasiregularly extends to Q by

- a degree d_0 quasiconformal covering $F|A(\beta_{0,1}, \gamma_{0,1}) : A(\beta_{0,1}, \gamma_{0,1}) \to A(\beta_1, \gamma_1)$
- a degree d_3 quasiconformal covering $F|A(\gamma_{3,1}, \beta_{3,1}) : A(\gamma_{3,1}, \beta_{3,1}) \to A(\gamma_1, \beta_1)$
- a degree d_3 quasiconformal covering $F|A(\beta_{3,0}, \gamma_{3,0}) : A(\beta_{3,0}, \gamma_{3,0}) \to A(\beta_0, \gamma_0)$
- a degree d_3 quasiconformal covering $F|A(\delta^+_{3,c}, \delta^-_{3,c}) : A(\delta^+_{3,c}, \delta^-_{3,c}) \to A(\delta^+_c, \delta^-_c)$

In particular, we have $F(Q) = A(\beta_1, \gamma_1) \cup A(\beta_0, \gamma_0) \cup A(\delta^+_c, \delta^-_c)$ (see figure 9 to follow the continuation of the proof).

Now denote by $\beta_{1,2}$ the preimage of β_2 in $B(\widehat{z_1})$ under $\widehat{f}$ (thus under F) and similarly by $\beta^-_{2,3}$ the preimage of β^-_3 in $B(\widehat{z_2})$ (see Figure 5). Moreover denote by $D(\beta_{1,2})$ (respectively by $D(\beta^-_{2,3})$) the open disk bounded by $\beta_{1,2}$ (respectively by $\beta^-_{2,3}$) and containing $\widehat{z_1}$ (respectively $\widehat{z_2}$). Finally let E be the union $D(\beta_{1,2}) \cup D(\beta^-_{2,3}) \cup D(\delta^-_{3,c}) \cup A(\gamma_{0,1}, \gamma_{3,1})$ (see Figure 9).

At first remark that E is an open subset of $H = D(\beta_{0,1}) \cup A(\gamma_{0,1}, \gamma_{3,1}) \cup A(\beta_{3,1}, \beta_{3,0}) \cup A(\gamma_{3,0}, \delta^+_{3,c}) \cup D(\delta^-_{3,c})$. Indeed we have $D(\beta_{1,2}) \cup D(\beta^-_{2,3}) \subset D(\beta_{0,1})$ from definition of $\beta_{0,1}$.

Moreover, it follows from the definition of F on H that $F(E) = D(\beta_2) \cup D(\beta^-_3) \cup D(\delta^-_c) \cup D(\gamma_1)$ where $D(\beta_2)$ denotes the open disk bounded by β_2 and containing $\widehat{z_2}$, and $D(\beta^-_3) = D(\delta^-_{3,c})$ denotes the open disk bounded by $\beta^-_3 = \delta^-_{3,c}$ and containing $\widehat{z_0}$.

Furthermore, according to the whole construction, we have

- from Lemma 3 and definition of γ_1: $\overline{A(\beta_1, \gamma_1)} \cup \overline{D(\gamma_1)} \subset D(\beta_{1,2})$ and $\overline{D(\beta_2)} \subset D(\beta^-_{2,3})$
- from definitions of δ^-_c, δ^+_c and γ_0: $\overline{A(\delta^+_c, \delta^-_c)} \cup \overline{D(\delta^-_c)} \subset A(\gamma_0, \gamma_{3,1}) \subset A(\gamma_{0,1}, \gamma_{3,1})$
- from definition of γ_0 and recaling $\beta_0 = \gamma_{0,1}$: $A(\beta_0, \gamma_0) \subset A(\gamma_{0,1}, \gamma_{3,1})$

Putting everything together gives the following diagram in which the arrows $\xrightarrow{F}$ stand for images under F, $\xrightarrow{\subset}$ stand for inclusions, $\xrightarrow{\overline{\subset}}$ stand for compact inclusions (namely $A \xrightarrow{\overline{\subset}} B$ if and only if $\overline{A} \subset B$) and $\xrightarrow{=}$ stands for equality.

$$
\begin{tikzcd}[column sep=small]
Q \arrow[d, "F"] & = & A(\beta_{0,1},\gamma_{0,1}) \arrow[d, "F"] & \cup & A(\gamma_{3,1},\beta_{3,1}) \arrow[dll, "F"'] & \cup & A(\beta_{3,0},\gamma_{3,0}) \arrow[dll, "F"'] & \cup & A(\delta^+_{3,c},\delta^-_{3,c}) \arrow[dll, "F"'] & & \\
F(Q) \arrow[d, "\subset"] & = & A(\beta_1,\gamma_1) \arrow[d, "\overline{\subset}"] & \cup & A(\beta_0,\gamma_0) \arrow[drrrr, "\subset"] & \cup & A(\delta^+_c,\delta^-_c) \arrow[drr, "\overline{\subset}"] & & & & \\
E \arrow[d, "F"] & = & D(\beta_{1,2}) \arrow[drr, "F"] & \cup & D(\beta^-_{2,3}) \arrow[drr, "F"] & \cup & D(\delta^-_{3,c}) \arrow[drr, "F"] & \cup & A(\gamma_{0,1},\gamma_{3,1}) \arrow[dllllll, "F"] & & \\
F(E) \arrow[d, "\subset"] & = & D(\gamma_1) \arrow[d, "\overline{\subset}"] & \cup & D(\beta_2) \arrow[d, "\overline{\subset}"] & \cup & D(\beta^-_3) \arrow[d, "="] & \cup & D(\delta^-_c) \arrow[d, "\overline{\subset}"] & & \\
E \arrow[d, "\subset"] & = & D(\beta_{1,2}) \arrow[d, "\overline{\subset}"] & \cup & D(\beta^-_{2,3}) \arrow[dll, "\overline{\subset}"'] & \cup & D(\delta^-_{3,c}) \arrow[drrrr, "="] & \cup & A(\gamma_{0,1},\gamma_{3,1}) \arrow[dllll, "="'] & & \\
H & = & D(\beta_{0,1}) & \cup & A(\gamma_{0,1},\gamma_{3,1}) & \cup & A(\beta_{3,1},\beta_{3,0}) & \cup & A(\gamma_{3,0},\delta^+_{3,c}) & \cup & D(\delta^-_{3,c})
\end{tikzcd}
$$

In particular, we deduce that $F(E) \subset E$ and $F^2(Q) \subset E \subset H$. Furthermore, following compact inclusions, it turns out that $F^2(\overline{Q}) \subset E$. $\square$

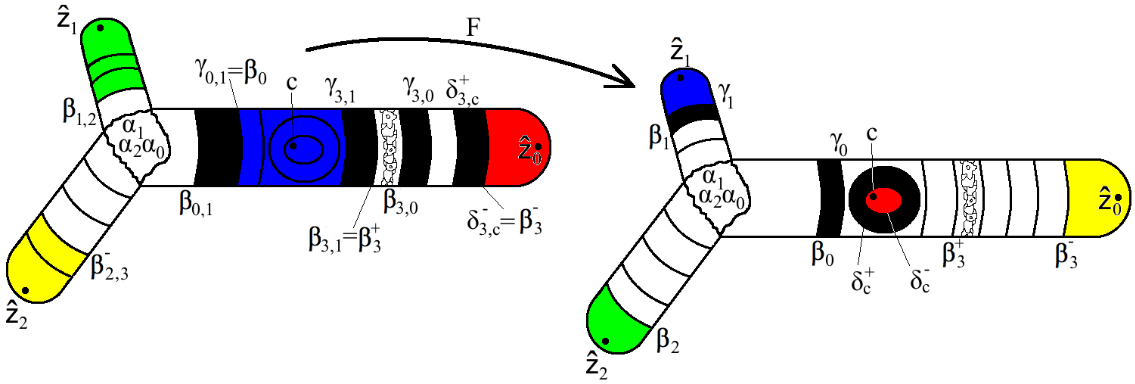


FIGURE 9: The maps F coming from Lemma 11. On the left topological sphere, the black area stands for Q and the colored area stands for E. On the right topological sphere, the black area stands for $F(Q)$ and the colored area stands for $F(E)$.

Now we have a quasiregular map F from the Riemann sphere to itself whose dynamics follows that one of the dynamical tree $P : \mathcal{H}_P \to \mathcal{H}_P$ (see Figure 4). We need to find a holomorphic map f conjugated to F so that f follows the same dynamics as well (f should satisfy the requirements of Theorem 3 and Theorem 4). In other words, we need to find a

complex structure on the Riemann sphere making F holomorphic. To do so, we will apply the Shishikura's fundamental lemma for quasiconformal surgery (stated for the first time in [Shi87]) that we recall below.

Lemma 12 (Shishikura's fundamental lemma for quasiconformal surgery)**.** *Let $G : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a quasiregular map. Assume there are an open set $E \subset \widehat{\mathbb{C}}$ and an integer $N \geqslant 0$ which satisfy the following conditions:*

- *$G(E) \subset E$*
- *G is holomorphic on E*
- *G is holomorphic on an open set containing $\widehat{\mathbb{C}} - G^{-N}(E)$*

Then there exists an almost complex structure μ on $\widehat{\mathbb{C}}$ which is G-invariant. In particular, if $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is the quasiconformal map of dilatation μ given by the measurable Riemann's mapping theorem then the map $\varphi \circ G \circ \varphi^{-1}$ is holomorphic.

The result stated in [Shi87] is a little more general but it easily implies the more explicit statement of Lemma 12 (we refer the readers to [Shi87] for more details and for a proof).

Here our map F satisfies the three assumptions (indeed F is holomorphic on H hence on $E \subset H$ and Lemma 11 implies that $\widehat{\mathbb{C}} - F^{-2}(E) \subset \widehat{\mathbb{C}} - \overline{Q} = H$), so applying Lemma 12 gives a holomorphic map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ quasiconformally conjugated to $F : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ as desired.

Lemma 13. *The rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ obtained above is of degree $\widehat{d} + d_3$ and has a super-attracting cycle $\{z_0, z_1, z_2, z_3\}$ of period 4 which is accumulated by every critical orbit. In particular, f is hyperbolic.*

Proof. Since f is quasiconformally conjugated to F, the critical points of f are images under a quasiconformal map φ of the critical points of F with same multiplicities. More precisely, the critical points of f are:

- $z_1 = \varphi(\widehat{z_1}) \in \varphi(D(\beta_{1,2})) \subset \varphi(E)$ with multiplicity $d_1 - 1$
- $z_2 = \varphi(\widehat{z_2}) \in \varphi(D(\beta_{2,3}^-)) \subset \varphi(E)$ with multiplicity $d_2 - 1$
- $d_0 + d_3$ critical points counted with multiplicity in $\varphi(A(\gamma_{0,1}, \gamma_{3,1})) \subset \varphi(E)$, which one of them is given by $z_0 = \varphi(c)$
- $z_3 = \varphi(\widehat{z_0}) \in \varphi(D(\delta_{3,c}^-)) \subset \varphi(E)$ with multiplicity $d_3 - 1$

According to the Riemann-Hurwitz formula, it follows that

$$2\deg(f) - 2 = (d_1 - 1) + (d_2 - 1) + (d_0 + d_3) + (d_3 - 1) = (d_0 + d_1 + d_2 - 1) + 2d_3 - 2$$

Therefore

$$\deg(f) = \frac{1}{2}(d_0 + d_1 + d_2 - 1) + d_3 = \widehat{d} + d_3$$

Notice that $\{z_0, z_1, z_2, z_3\}$ forms a super-attracting cycle of period 4. Moreover every critical points of f lies in the forward invariant open set $\varphi(E)$, namely an union of 4 connected components each containing one point of $\{z_0, z_1, z_2, z_3\}$. Consequently, every critical orbit accumulates this super-attracting cycle. $\square$

4 Properties

The aim of this section is to achieve the proofs of Theorem 3 and Theorem 4. More precisely we are going to show that the rational map f constructed in the previous section satisfies all the requirements of these two theorems. Section 4.1 focuses on the dynamical properties of f (stated in Theorem 3), then Section 4.2 deals with the topological properties of the Julia component of f (stated in Theorem 4).

In order to lighten notations, we forget the quasiconformal map φ provided by Lemma 12 to denote the image under φ of any set introduced in the previous section (equivalently speaking, we act as if the quasiregular map F constructed in the previous section is actually holomorphic).

4.1 Exchanging dynamics

Consider the following pairwise disjoint collection of open annuli (see Figure 10).

$$A_0 = A(\alpha_0, \beta_0), \quad A_1 = A(\alpha_1, \beta_1), \quad A_2 = A(\alpha_2, \beta_2), \quad \text{and} \quad A_3 = A(\beta_3^+, \beta_3^-)$$

Then, consider the connected components of the preimage under f of this collection which are each contained as essential subannulus in one of these annuli, namely:

- $A_{0,1} = A(\alpha_0, \beta_{0,1})$
- $A_{1,2} = A(\alpha_1, \beta_{1,2})$
- $A_{2,0} = A(\alpha_2, \beta_{2,0})$ where $\beta_{2,0}$ is the preimage of β_0 in $B(\widehat{z_2})$ (see Figure 5)
- $A_{2,3} = A(\beta_{2,3}^+, \beta_{2,3}^-)$ where $\beta_{2,3}^+$ is the preimage of β_3^+ in $B(\widehat{z_2})$ (see Figure 5)
- $A_{3,0} = A(\alpha_{3,0}, \beta_{3,0})$ where $\alpha_{3,0}$ is the preimage of α_0 in $A(\beta_{3,1}, \beta_{3,0})$ (see Lemma 7)
- $A_{3,1} = A(\beta_{3,1}, \alpha_{3,1})$ where $\alpha_{3,1}$ is the preimage of α_1 in $A(\beta_{3,1}, \beta_{3,0})$ (see Lemma 7)

Notice that the notation is chosen so that each $A_{i,j}$ is contained as essential subannulus in A_i, and $f_{|A_{i,j}} : A_{i,j} \to A_j$ is a degree d_i covering. Remark that some connected components of $f^{-1}(A_3)$ are included in A_3 as well (from Lemma 9, see Figure 8), but none of them is contained in A_3 as essential subannulus.

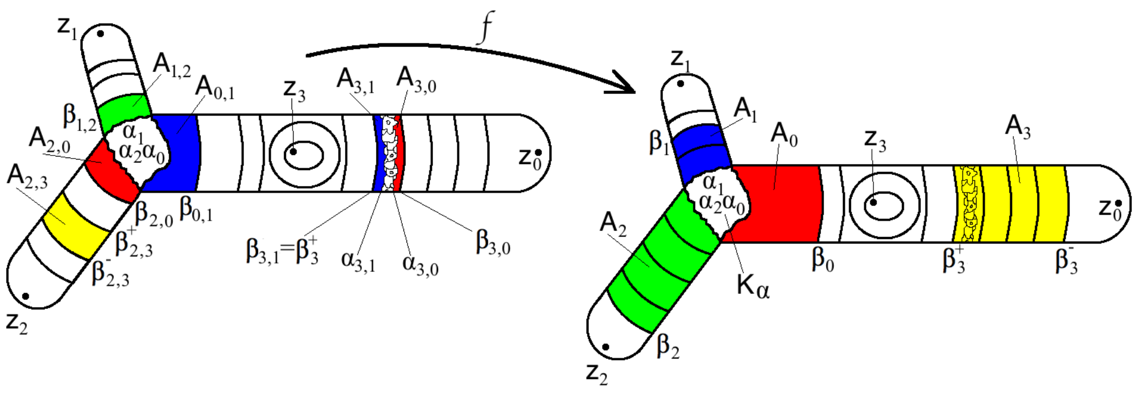


FIGURE 10: The various annuli considered to encode the exchanging dynamics.

Denote by $\mathcal{A}$ the collection of all connected components of the non-escaping set induced by $f_{|U} : U \to A_0 \cup A_1 \cup A_2 \cup A_3$ on the union of subannuli $U = A_{0,1} \cup A_{1,2} \cup A_{2,0} \cup A_{2,3} \cup A_{3,0} \cup A_{3,1}$. More precisely:

$$\mathcal{A} = \Big\{ J \text{ connected component of } \{z \in U \,/\, \forall n \geqslant 0,\ f^n(z) \in U\} \Big\}$$

Let J_α be the continuum in $\widehat{\mathbb{C}}$ which corresponds to the Julia set $J(\widehat{f})$ of $\widehat{f}$ (more precisely, J_α is the image of $J(\widehat{f})$ under the quasiconformal map φ provided by Lemma 12). Remark that J_α is fixed under iteration of f and J_α intersects $\overline{U}$ (along $\alpha_0 \cup \alpha_1 \cup \alpha_2$). Denote by $\mathcal{A}_\alpha$ the collection of all continua which are eventually mapped onto J_α and whose every iterate intersects $\overline{U}$.

$$\mathcal{A}_\alpha = \Big\{ J \text{ connected component of } \bigcup_{n \geqslant 0} f^{-n}(J_\alpha) \text{ such that } \forall n \geqslant 0,\ f^n(J) \cap \overline{U} \neq \emptyset \Big\}$$

Finally , denote by $\mathcal{A}^\star$ the union $\mathcal{A} \cup \mathcal{A}_\alpha$. As collection of pairwise disjoint continua, $\mathcal{A}^\star$ is endowed with the topology coming from the usual distance between continua on the Riemann sphere $\widehat{\mathbb{C}}$ (equipped with the spherical metric). It turns out that f induced a topological dynamical system on $\mathcal{A}^\star$. This dynamical system may be encoded by the weighted Hubbard tree $(\mathcal{H}_P, w)$ (see Section 2.2) as it is shown in the following lemma.

Lemma 14. *There exists a homeomorphism $h : \mathcal{A}^\star \to \mathcal{J}(\mathcal{H}_P)$ such that the following diagram commutes*

$$\begin{tikzcd} \mathcal{A}^\star \arrow[r, "f"] \arrow[d, "h"'] & \mathcal{A}^\star \arrow[d, "h"] \\ \mathcal{J}(\mathcal{H}_P) \arrow[r, "P"'] & \mathcal{J}(\mathcal{H}_P) \end{tikzcd}$$

Moreover for every $J \in \mathcal{A}$, the restriction map $f|_J$ is of degree $w(e_k) = d_k$ where e_k is the edge of $\mathcal{H}_P$ which contains $h(J)$.

Proof. At first, remark there is a subannulus $A_{i,j}$ for some $i, j \in \{0, 1, 2, 3\}$ if and only if the (i, j)-entry of the transition matrix $M = (m_{i,j})_{i,j \in \{0,1,2,3\}}$ is non-zero (see Definition 1). Indeed, recall that the transition matrix is

$$M = \begin{pmatrix} 0 & \frac{1}{d_0} & 0 & 0 \\ 0 & 0 & \frac{1}{d_1} & 0 \\ \frac{1}{d_2} & 0 & 0 & \frac{1}{d_2} \\ \frac{1}{d_3} & \frac{1}{d_3} & 0 & 0 \end{pmatrix}$$

According to this remark, we introduce the subshift of finite type (Σ, σ) associated to the transition matrix M, namely the restriction of the 4-to-1 shift map on the subset of all infinite sequences of digits in $\{0, 1, 2, 3\}$ such that every adjacent pair of entries lies in $\{(0, 1), (1, 2), (2, 0), (2, 3), (3, 0), (3, 1)\}$.

$$\Sigma = \Big\{ s = (s_0, s_1, s_2, \dots) \in \{0, 1, 2, 3\}^{\mathbb{N}} \,/\, \forall k \geqslant 0,\ m_{s_k, s_{k+1}} \neq 0 \Big\}$$

$$\sigma : \Sigma \to \Sigma,\, s = (s_0, s_1, s_2, \dots) \mapsto \sigma(s) = (s_1, s_2, s_3, \dots)$$

Σ is endowed with the topology coming from the following distance, making it a Cantor set.

$$\forall s, s' \in \Sigma,\ d(s, s') = \sum_{k \geqslant 0} \frac{|s_k - s'_k|}{4^k}$$

Let S_α be the subset of Σ of three infinite sequences of repeating $0, 1, 2$ digits.

$$S_\alpha = \Big\{(0,1,2,0,1,2,0,1,2,\dots),(1,2,0,1,2,0,1,2,0,\dots),(2,0,1,2,0,1,2,0,1,\dots)\Big\}$$

We shall identify these three sequences in Σ, and similarly every subset of sequences which are eventually mapped in S_α after the same itinerary under σ. More precisely, let $\sim$ be the equivalence relation on Σ defined by

$$\forall s, s' \in \Sigma,\ s \sim s' \iff \exists n \geqslant 0 \,/ \left\{ \begin{array}{l} \forall k \in \{0, 1, \dots, n\},\ s_k = s'_k \\ \sigma^n(s), \sigma^n(s') \in S_\alpha \end{array} \right.$$

and let $\Sigma^\star$ be the topological quotient space $\Sigma/\sim$. Remark that $\Sigma^\star$ is a Cantor set as well for the quotient topology induced by $\sim$. Abusing notations, every equivalence class containing only one infinite sequence $s \in \Sigma$ which is not eventually mapped in S_α is still denoted by $s \in \Sigma^\star$, and the map induced by the shift map on $\Sigma^\star$ is still denoted by σ.

We are going to show that $(\mathcal{A}^\star, f)$ is topologically conjugated to $(\Sigma^\star, \sigma)$. To do so, consider the itinerary map $h_1 : \mathcal{A} \to \Sigma^\star$ defined as follows

$$\forall J \in \mathcal{A},\ h_1(J) = (s_0, s_1, s_2, \dots) \quad \text{with} \quad \forall k \geqslant 0,\ f^k(J) \subset A_{s_k}$$

This map is well defined and injective by definition of $\mathcal{A}$.

To prove that h_1 extends to a homeomorphism from $\mathcal{A}^\star$ to $\Sigma^\star$, we first define by induction for every $s = (s_0, s_1, s_2, \dots) \in \Sigma$ an infinite sequence of subannuli $(A_{s_0,s_1,\dots,s_n})_{n\geqslant 0}$ such that for every $n \geqslant 0$, $A_{s_0,s_1,\dots,s_n}$ is contained in A_{s_0} as essential subannulus, and $f_{|A_{s_0,s_1,\dots,s_n}} : A_{s_0,s_1,\dots,s_n} \to A_{s_1,s_2,\dots,s_n}$ is a degree d_{s_0} covering. Denote by $A_s = A_{s_0,s_1,s_2,\dots}$ the limit set $\bigcap_{n\geqslant 0} \overline{A_{s_0,s_1,\dots,s_n}}$ which is a continuum.

If s is not eventually mapped in S_α, then $\overline{A_{s_0,s_1,\dots,s_n}}$ is contained in $U = A_{0,1} \cup A_{1,2} \cup A_{2,0} \cup A_{2,3} \cup A_{3,0} \cup A_{3,1}$ for every $n \geqslant 0$ large enough and thus A_s is a connected component of the non-escaping set, that is an element of $\mathcal{A}$. Moreover, $h_1(A_s) = s$ holds from the definition of the itinerary map h_1.

On the contrary, if s is in S_α, then A_s is either α_0, α_1 or α_2, and in particular A_s is contained in J_α. More generally, if s is eventually mapped in S_α, then A_s is contained in a continuum J which is eventually mapped onto J_α, that is an element of $\mathcal{A}_\alpha$. Moreover, for every $s' \in \Sigma$ such that $s' \sim s$, $A_{s'}$ is contained in the same continuum $J \in \mathcal{A}_\alpha$.

Therefore h_1 extends to a bijective map from $\mathcal{A}^\star$ to $\Sigma^\star$, by associating to $J \in \mathcal{A}_\alpha$ the equivalence class $h_1(J) \in \Sigma^*$ of the itinerary $s = (s_0 s_1, s_2, \dots) \in \Sigma$ of any subcontinuum in J which is eventually mapped into $\alpha_0 \cup \alpha_1 \cup \alpha_2$. Furthermore, this extension is actually a conjugation between f and σ.

$$\forall J \in \mathcal{A}^\star,\ h_1(f(J)) = \sigma(h_1(J))$$

It remains to prove the continuity. Fix $J \in \mathcal{A}^\star$ and let $s = (s_0, s_1, s_2, \dots)\Sigma$ be a class representative of $h_1(J)$. Let J' be another element of $\mathcal{A}^\star$ such that some class representative $s' = (s'_0, s'_1, s'_2, \dots) \in \Sigma$ of $h_1(J')$ is arbitrary close to s. That implies the first n digits of s and s' coincide for arbitrary large $n \geqslant 0$. In particular, A_s and $A_{s'}$ are contained in $\overline{A_{s_0,s_1,\dots,s_n}}$. Remark that $f^n_{|A_{s_0,s_1,\dots,s_n}} : A_{s_0,s_1,\dots,s_n} \to A_{s_n}$ is a covering of degree $d_{s_0} d_{s_1} \dots d_{s_{n-1}}$ tending to infinity with n (since assumption (H2) implies that at least three of weights d_0, d_1, d_2, d_3 are $\geqslant 2$, see Definition 1). Therefore A_s and $A_{s'}$ are contained in an annulus of arbitrary small modulus. Then, using extremal length, it follows that $A_s \subset J$ and $A_{s'} \subset J'$ are arbitrary close, hence J and J' are arbitrary close in $\mathcal{A}^\star$. Consequently h_1^{-1} is continuous. The continuity of h_1 follows from a similar argument.

Similarly, we can show that $(\mathcal{J}(\mathcal{H}_P), P)$ is topologically conjugated to $(\Sigma^\star, \sigma)$ by a homeomorphism $h_2 : \mathcal{J}(\mathcal{H}_P) \to \Sigma^\star$. Indeed recall that the Hubbard tree $\mathcal{H}_P$ is described by a set of four edges e_0, e_1, e_2, e_3 where P acts as follows (see Section 2.2)

$$\begin{cases} P(e_0) = e_1 \\ P(e_1) = e_2 \\ P(e_2) = e_0 \cup e_3 \\ P(e_3) = e_0 \cup e_1 \end{cases}$$

Thus, we may find a collection of four connected open subsets I_0, I_1, I_2, I_3 included in e_0, e_1, e_2, e_3 respectively, together with a collection of six connected open subsets $I_{0,1}, I_{1,2}, I_{2,0}, I_{2,3}, I_{3,0}, I_{3,1}$ such that:

- each $I_{i,j}$ is contained in I_i and $P_{|I_{i,j}} : I_{i,j} \to I_j$ is a homeomorphism
- and $\mathcal{J}(\mathcal{H}_P) = \Big\{z \in V \,/\, \forall n \geqslant 0,\ P^n(z) \in V\Big\} \cup \Big\{z \text{ point in } \bigcup_{n \geqslant 0} P^{-n}(\alpha) \cap \overline{V}\Big\}$ where $V = I_{0,1} \cup I_{1,2} \cup I_{2,0} \cup I_{2,3} \cup I_{3,0} \cup I_{3,1}$

Consequently, we can show as above that the itinerary map $h_2 : \{z \in V \,/\, \forall n \geqslant 0,\ P^n(z) \in V\} \to \Sigma^\star$ extends to a homeomorphism from $\mathcal{J}(\mathcal{H}_P)$ to $\Sigma^\star$ which conjugates the dynamics of P and σ.

Finally, taking $h = h_2^{-1} \circ h_1$ concludes the proof. □

Remark that the proof of Theorem 3 is almost completed. Indeed point **(i)** comes from Lemma 13 while points **(ii)** and **(iii)** follows from Lemma 14 (since $\mathcal{A}$ is, by definition, the set of continua J in $\mathcal{A}^\star$ such that J is not eventually mapped under iterations to the fixed continuum J_α, or equivalently, such that $h(J)$ is not eventually mapped under iterations to the fixed branching point α). It only remains to prove that $\mathcal{A}^\star$ is actually the set of all critically separating Julia components of f.

Lemma 15. *The following equality of sets holds.*

$$\mathcal{A}^\star = \mathcal{J}_{crit}(f)$$

Proof. Recall that the postcritical set is contained in the forward invariant set $E = D(\beta_{1,2}) \cup D(\beta_{2,3}^-) \cup D(\delta_{3,c}^-) \cup A(\gamma_{0,1}, \gamma_{3,1})$ (see Lemma 11 and Figure 9) and each point of the super-attracting cycle $\{z_0, z_1, z_2, z_3\}$ lies in a different connected component of E. In particular $J(f)$ is the set of all points whose orbit remains in $\widehat{\mathbb{C}} - E = \overline{A_0} \cup \overline{A_1} \cup \overline{A_2} \cup \overline{A_3} \cup K_\alpha$ where K_α is the complement in $\widehat{\mathbb{C}}$ of $B(\widehat{z_0}) \cup B(\widehat{z_1}) \cup B(\widehat{z_2})$ (see Figure 10).

It follows that every element J in $\mathcal{A}$ is a Julia component. Moreover J is critically separating as limit set of nested essential subannuli which separate each the super-attracting cycle $\{z_0, z_1, z_2, z_3\}$ (see proof of Lemma 14). Therefore $\mathcal{A} \subset \mathcal{J}_{\text{crit}}(f)$.

Similarly, every element J in $\mathcal{A}_\alpha$ is a Julia component. Moreover recall that J intersects $\overline{U}$ along a limit set of nested essential subannuli which separate each the super-attracting cycle $\{z_0, z_1, z_2, z_3\}$ (see proof of Lemma 14). Therefore $\mathcal{A}_\alpha \subset \mathcal{J}_{\text{crit}}(f)$ and $\mathcal{A}^\star = \mathcal{A} \cup \mathcal{A}_\alpha \subset \mathcal{J}_{\text{crit}}(f)$.

Conversely, let J be a critically separating Julia component of f. Remark that J is not contained in $K_\alpha - J_\alpha$. Indeed, recall that every connected component of $\widehat{\mathbb{C}} - J_\alpha$ is simply connected (see Lemma 2) and that $\partial K_\alpha = \alpha_0 \cup \alpha_1 \cup \alpha_2 \subset J_\alpha$, therefore every connected compact subset of any connected component of $K_\alpha - J_\alpha$ does not separate the postcritical points. Consequently either J is $J_\alpha \in \mathcal{A}_\alpha \subset \mathcal{A}^\star$ or $f^n(J)$ stays in $\overline{A_0} \cup \overline{A_1} \cup \overline{A_2} \cup \overline{A_3}$ for every $n \geqslant 0$. Assume that J is not J_α.

Recall that every connected component of the preimage under f of $\overline{A_0} \cup \overline{A_1} \cup \overline{A_2} \cup \overline{A_3}$ which is contained in this compact union, is contained either in $\overline{U}$ or in some connected component of $\overline{f^{-1}(A_3)}$ included in A_3 (from Lemma 9, see Figure 8), says $\overline{A'_{3,3}}$. However every $A'_{3,3}$ is not contained in A_3 as essential subannulus, and hence does not separate the postcritical points. In particular J is not contained in any $\overline{A'_{3,3}}$. Furthermore, J can not eventually fall in some $\overline{A'_{3,3}}$ after some iterations of f, otherwise $f^n(J)$ would not be critically separating for some $n \geqslant 0$ contradicting the fact that J is critically separating. It follows that $f^n(J)$ stays in $\overline{U}$ for every $n \geqslant 0$ and hence $J \in \mathcal{A}^\star$ that concludes the proof. □

4.2 Topology of buried Julia components

Existence of each of the three types of buried Julia components which occurs in $J(f)$ is shown in this section, that proofs Theorem 4.

Lemma 16 (Point type buried Julia components)**.** *There exist uncountably many buried Julia components in $J(f)$ which are points.*

Proof. Let $A'_{3,3} = A(\beta^+_{3,3}, \beta^-_{3,3})$ be a connected component of $f^{-1}(A_3)$ contained in $A_3 = A(\beta^+_3, \beta^-_3)$ (from Lemma 9, see Figure 8) where $\beta^+_{3,3}, \beta^-_{3,3}$ are preimages of β^+_3, β^-_3 respectively. Recall that $A'_{3,3}$ is not contained in A_3 as essential subannulus. In particular, the connected component of $\widehat{\mathbb{C}} - \beta^+_{3,3}$ containing $A'_{3,3}$ is an open disk $D(\beta^+_{3,3})$ contained in A_3 and such that $f_{|D(\beta_{3,3})} : D(\beta^+_{3,3}) \to D$ is a homeomorphism where $D = D(\beta^+_3)$ is the open disk bounded by β^+_3 and containing A_3.

Using notations coming from the proof of Lemma 14, consider the subannulus $A_{3,0,1,2,3}$ contained in A_3 as essential subannulus and such that $f^4_{|A_{3,0,1,2,3}} : A_{3,0,1,2,3} \to A_3$ is a degree $d_3 d_0 d_1 d_2$ covering. Since assumption (H2) implies that at least three of weights d_0, d_1, d_2, d_3 are $\geqslant 2$ (see Definition 1), it follows that this degree is $\geqslant 2$ and hence, there are at least 2 disjoint preimages under $f^4_{|A_{3,0,1,2,3}}$ of $D(\beta^+_{3,3})$ in $A_{3,0,1,2,3} \subset A_3 \subset D$, says D_0 and D_1.

Finally we have two disjoint open disks D_0 and D_1 in D such that $f^5_{|D_0} : D_0 \to D$ and $f^5_{|D_1} : D_1 \to D$ are homeomorphims. It is then a classical exercise to prove that the non-escaping set

$$\mathcal{D} = \{z \in D_0 \cup D_1 \,/\, \forall n \geqslant 0, (f^5)^n \in D_0 \cup D_1\}$$

is a Cantor set homeomorphic to the space of all sequences of two digits $\Sigma_2 = \{0,1\}^{\mathbb{N}}$. In particular, $\mathcal{D}$ contains uncountably many points. Furthermore every point in $\mathcal{D}$ is a buried point in $J(f)$ since $A_3 \subset D$ contains infinitely many critically separating Julia components. □

Lemma 17 (Circle type buried Julia components)**.** *There exist uncountably many buried Julia components in $J(f)$ which are wandering Jordan curves.*

Proof. This is mostly a consequence of the main result in [PT00] claiming that every wandering Julia component of a geometrically finite rational map is either a point or a Jordan curve. Here our map f is hyperbolic (from Lemma 13) therefore every wandering Julia component in $\mathcal{J}_{\text{crit}}(f)$ must be a Jordan curve (since a point is obviously not critically separating). Moreover, according to the proof of Lemma 14, the set of wandering Julia components in $\mathcal{J}_{\text{crit}}(f)$ exactly corresponds to the set of all the infinite sequences in $\Sigma^\star$ which are not eventually periodic. In particular, there are uncountably many such Julia components. Finally, uncountably many of them must be buried since the Fatou set only has countably many Fatou domains and each of them only has countably many Jordan curves as connected components of its boundary. □

Lemma 18 (Complex type buried Julia components). *J_α and all its countably many preimages, are buried Julia components in $J(f)$.*

Proof. Coming back to the proof of Lemma 14, remark that each infinite sequence in S_α is not isolated in Σ. Therefore, α_k has no intersection with the boundary of any Fatou domain contained in $B(\widehat{z_k})$ for every $k \in \{0,1,2\}$. It remains to show that J_α has no intersection with the boundary of any Fatou domain in $K_\alpha = \widehat{\mathbb{C}} - (B(\widehat{z_0}) \cup B(\widehat{z_1}) \cup B(\widehat{z_2}))$. Recall that every connected component of $K_\alpha - J_\alpha$, that is a connected component of $\widehat{\mathbb{C}} - J_\alpha$, is eventually mapped under iterations onto $B(\widehat{z_k})$ for some $k \in \{0,1,2\}$ (since f is defined to be $\widehat{f}$ on $K_\alpha \subset D(\beta_{0,1})$). By continuity of f, it follows that J_α has no intersection with the boundary of any Fatou domain contained in any connected component of $K_\alpha - J_\alpha$. Consequently J_α is buried. The same holds as well for every preimage of J_α by continuity of f. □

5 Explicit formula in the cubic case

In this last section, we proof Theorem 1 stated in Introduction. Firstly we show that a particular choice of the weight function w gives a rational map of degree 3 (in Lemma 19). Then we compute an explicit formula for this particular example.

Lemma 19. *The following weight function on the set of edges of $\mathcal{H}_P$*

$$(d_0, d_1, d_2, d_3) = (1, 2, 2, 1)$$

satisfies assumptions (H1) and (H2) from Theorem 3 and Theorem 4. In particular there are some rational maps of degree 3 whose Julia set contains buried Julia components of several types:

point type: *uncountably many points*

circle type: *uncountably many Jordan curves*

complex type: *countably many preimages of a fixed Julia component which is quasiconformally homeomorphic to the connected Julia set of $\widehat{f} : z \mapsto \frac{1}{(z-1)^2}$*

Proof. Assumption (H1) is obviously satisfied, indeed

$$\widehat{d} = \frac{1}{2}(d_0 + d_1 + d_2 - 1) = \frac{1}{2}(1 + 2 + 2 - 1) = 2 = \max\{d_0, d_1, d_2\}$$

For assumption (H2), the transition matrix (see Definition 1) for this choice of weight function is given by

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

and an easy computation shows that $\lambda(\mathcal{H}_P, w)$ is the largest root of $X^4 - \frac{1}{2}X - \frac{1}{4}$ that is $\lambda(\mathcal{H}_P, w) \approx 0.918 < 1$.

Applying Theorem 3 and Theorem 4 gives a rational map of degree $\widehat{d} + d_3 = 2 + 1 = 3$.

Furthermore, recall that the rational map $\widehat{f}$ which appears in Theorem 4 is of degree $\widehat{d} = 2$ and has only one critical orbit which is a super-attracting cycle $\{\widehat{z_0}, \widehat{z_1}, \widehat{z_2}\}$ of period 3 such that the local degrees of $\widehat{f}$ at $\widehat{z_0}$, $\widehat{z_1}$ and $\widehat{z_2}$ are $d_0 = 1$, $d_1 = 2$ and $d_2 = 2$ respectively. Up to

conjugation by a Möbius map, we may assume that $\widehat{z_0} = 0$, $\widehat{z_1} = 1$ and $\widehat{z_2} = \infty$. It turns out that there is then only one such quadratic rational map which is $\widehat{f} : z \mapsto \frac{1}{(z-1)^2}$.

$$\begin{tikzcd} \widehat{z_0} = 0 \arrow[r, "1:1"] & \widehat{z_1} = 1 \arrow[r, "2:1"] & \widehat{z_2} = \infty \arrow[ll, bend right, "2:1"'] \end{tikzcd}$$

□

Remark that this choice of weight function is the only one which gives a degree 3 and which satisfies assumptions (H1) and (H2).

The construction by quasiconformal surgery detailed in Section 3 does not provide an algebraic formula for the rational map f in Theorem 3 and Theorem 4. Furthermore the degree $\widehat{d} + d_3$ of f increases quickly with the weight function w so the algebraic relations behind are complicated to study. However the particular rational map of degree 3 coming from Lemma 19 is simple enough to allow a computation by hand of an algebraic formula.

Let f be a rational map coming from the construction detailed in Section 3 for the particular choice of weight function in Lemma 19 . Recall that the local degrees of f at z_1, z_2 and z_3 are $d_1 = 2$, $d_2 = 2$ and $d_3 = 1$ respectively. In particular, z_1 and z_2 are simple critical points. It remains $d_0 + d_3 = 1 + 1 = 2$ critical points counted with multiplicity coming from definition of f near z_0 (see Lemma 6), namely two simple critical points, one is z_0 by construction and the orbit of the other one accumulates the super-attracting cycle $\{z_0, z_1, z_2, z_3\}$.

Up to conjugation by a Möbius map, we assume that $z_1 = 1$, $z_2 = \infty$ and $z_3 = 0$. So 1 and ∞ are critical points whereas 0 is a singular point. In order to simplify notations, denote by λ the critical point z_0 (λ will be the parameter of our family) and by λ' the last critical point.

$$\begin{tikzcd} z_0 = \lambda \arrow[r, "2:1"] & z_1 = 1 \arrow[r, "2:1"] & z_2 = \infty \arrow[r, "2:1"] & z_3 = 0 \arrow[lll, bend right, "1:1"'] \\ \lambda' \arrow[r, "2:1"] & \ldots & & \end{tikzcd}$$

Since f is of degree 3, it is of the form

$$f : z \mapsto \frac{a_3 z^3 + a_2 z^2 + a_1 z + a_0}{b_3 z^3 + b_2 z^2 + b_1 z + b_0}$$

Since $z_1 = 1$ is mapped to $z_2 = \infty$ with a local degree 2, the denominator may factor as

$$f : z \mapsto \frac{a_3 z^3 + a_2 z^2 + a_1 z + a_0}{(z-1)^2 (b'_1 z + b'_0)}$$

We do likewise for $z_2 = \infty$ which is mapped to $z_3 = 0$ with a local degree 2.

$$f : z \mapsto \frac{a_1 z + a_0}{(z-1)^2 (b'_1 z + b'_0)}$$

Now use the fact that $z_3 = 0$ is mapped to $z_0 = \lambda$ to get

$$f : z \mapsto \frac{a_1 z + \lambda}{(z-1)^2 (b'_1 z + 1)} \tag{5}$$

It remains two informations coming from the fact that $z_0 = \lambda$ is mapped to $z_1 = 1$ with a local degree 2. Namely $f(\lambda) = 1$ and $f'(\lambda) = 0$ which lead to the two following equations satisfied by a_1 and b_1'.

$$\begin{cases} (\lambda-1)^2(\lambda b_1'+1) &= \lambda(a_1+1) \\ a_1(\lambda-1)^2(\lambda b_1'+1) &= \lambda(a_1+1)\Big[(3\lambda^2-4\lambda+1)b_1'+2(\lambda-1)\Big] \end{cases}$$

Remark that we may easily simplify the second equation by using the first one (luckily) to get

$$\begin{cases} (\lambda-1)^2(\lambda b_1'+1) &= \lambda(a_1+1) \\ a_1 &= (3\lambda^2-4\lambda+1)b_1'+2(\lambda-1) \end{cases}$$

or equivalently

$$\begin{cases} \lambda a_1 - \lambda(1-\lambda)^2 b_1' &= 1-3\lambda+\lambda^2 \\ a_1 - (1-\lambda)(1-3\lambda)b_1' &= -2+2\lambda \end{cases}$$

Solving this linear system of two equations gives:

$$\begin{cases} a_1 = \dfrac{(1-3\lambda)(1-3\lambda+\lambda^2)-\lambda(1-\lambda)(-2+2\lambda)}{\lambda(1-3\lambda)-\lambda(1-\lambda)} = \dfrac{1-4\lambda+6\lambda^2-\lambda^3}{-2\lambda^2} \\ b_1' = \dfrac{(1-3\lambda+\lambda^2)-\lambda(-2+2\lambda)}{-\lambda(1-\lambda)^2+\lambda(1-\lambda)(1-3\lambda)} = \dfrac{1-\lambda-\lambda^2}{-2\lambda^2(1-\lambda)} \end{cases}$$

Finally, putting these expressions in (5) leads to the following formula for f which depends on the parameter λ.

$$\boxed{f_\lambda : z \mapsto \frac{(1-\lambda)\Big[(1-4\lambda+6\lambda^2-\lambda^3)z-2\lambda^3\Big]}{(z-1)^2\Big[(1-\lambda-\lambda^2)z-2\lambda^2(1-\lambda)\Big]}}$$

Remark that $f_\lambda(z) = \frac{1}{(z-1)^2}(1-4\lambda+O_{\lambda\to 0}(\lambda^2))$ for every complex number z, thus f_λ is actually a particular perturbation of $f_0 = \widehat{f} : z \mapsto \frac{1}{(z-1)^2}$.

Some more computations provide an algebraic formula for the critical point λ', namely

$$\lambda' = -\frac{\lambda(1-6\lambda+11\lambda^2-10\lambda^3+5\lambda^4)}{(1-\lambda-\lambda^2)(1-4\lambda+6\lambda^2-\lambda^3)} = -\lambda + \underset{\lambda\to 0}{O}(\lambda^2)$$

According to the construction detailed in Section 3, there exist some choices of λ such that f_λ satisfies Theorem 1. Recall that the two critical points $z_0 = \lambda$ and $\lambda' \sim_{\lambda\to 0} -\lambda$ should lie in $B(\widehat{z_0})$ (see Section 3), and hence near $\widehat{z_0}$ which corresponds to $z_3 = 0$.

Indeed, we have:

$$f_\lambda(\lambda') = 1 - 8\lambda + \underset{\lambda\to 0}{O}(\lambda^2)$$

Furthermore, we can roughly prove for every $|\lambda| > 0$ small enough that

- the image under f_λ of a disk centered at $z_1 = 1$ and of radius of order $|\lambda|$ is contained in the complement of a disk centered at 0 (thus containing $z_2 = \infty$) and of radius of order $|\lambda|^{-2}$
- the image under f_λ of the complement of a disk centered at 0 (thus containing $z_2 = \infty$) and of radius of order $|\lambda|^{-2}$ is contained in a disk centered at $z_3 = 0$ and of radius of order $|\lambda|^4$

- the image under f_λ of a disk centered at $z_3 = 0$ and of radius of order $|\lambda|^4$ is contained in a disk centered at $z_0 = \lambda$ and of radius of order $|\lambda|^2$
- the image under f_λ of a disk centered at $z_0 = \lambda$ and of radius of order $|\lambda|^2$ is contained in a disk centered at $z_1 = 1$ and of radius of order $|\lambda|^3$

It turns out that the orbit of the critical point λ' accumulates the super-attracting cycle $\{z_0, z_1, z_2, z_3\}$ for every $|\lambda| > 0$ small enough. Consequently, we may encode the exchanging dynamics of Julia components of f_λ as it was explained in Section 4, proving that f_λ satisfies Theorem 1 for every $|\lambda| > 0$ small enough. Numerically, picking any parameter λ in the hyperbolic component centered at 0 of the parameter space for the family f_λ provides a Persian Carpet example in the dynamical plane (see Figure 11).

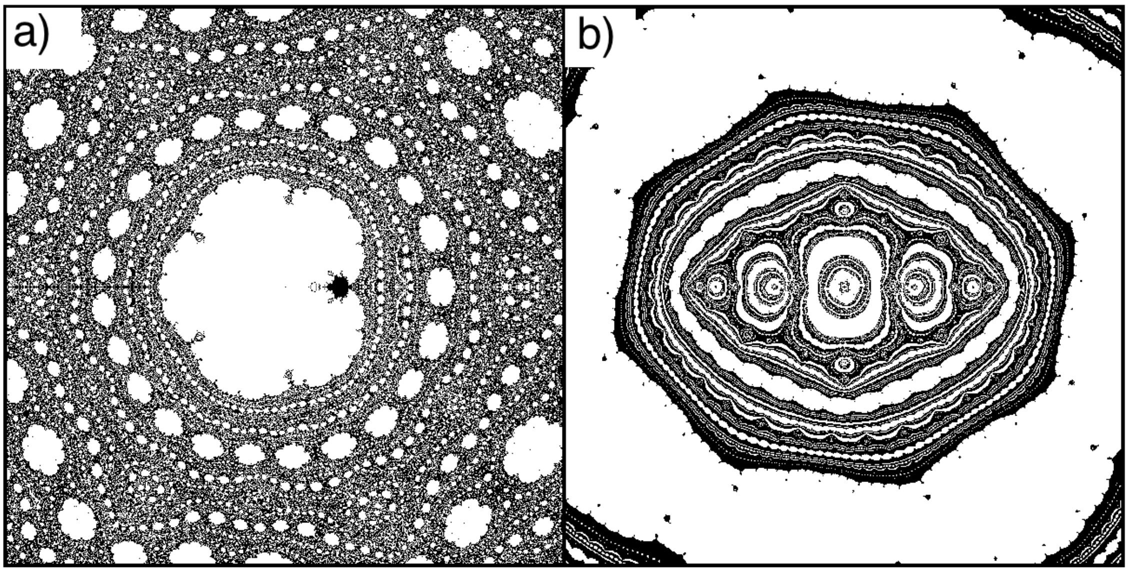


FIGURE 11: **a)** The bifurcation locus for the family f_λ with $|\lambda| \lessapprox 10^{-2}$
b) A Persian carpet: $J(f_\lambda)$ with $\lambda \approx 10^{-3}$

References

[Ahl10] Lars V. Ahlfors. *Conformal invariants.* AMS Chelsea Publishing, Providence, RI, 2010.

[BDGR08] Paul Blanchard, Robert L. Devaney, Antonio Garijo, and Elizabeth D. Russell. A generalized version of the McMullen domain. *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, 18(8):2309–2318, 2008.

[Bea91] Alan F. Beardon. *Iteration of rational functions*, volume 132 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, NY, 1991.

[CG93] Lennart Carleson and Theodore W. Gamelin. *Complex dynamics.* Universitext: Tracts in Mathematics. Springer-Verlag, New York, NY, 1993.

[CT11] Cui Guizhen and Tan Lei. A characterization of hyperbolic rational maps. *Inventiones Mathematicae*, 183(3):451–516, 2011.

[DH84] Adrien Douady and John H. Hubbard. *Étude dynamique des polynômes complexes. Partie I*, volume 84 of *Publications Mathématiques d'Orsay*. Université de Paris-Sud, Orsay, 1984.

[DHL+08] Robert L. Devaney, Matt Holzer, Daniel M. Look, Mónica Moreno Rocha, and David Uminsky. Singular perturbations of z^n. In *Transcendental dynamics and complex analysis*, volume 348 of *London Mathematical Society Lecture Note Series*, pages 111–137. Cambridge University Press, Cambridge, 2008.

[DM08] Robert L. Devaney and Sebastian M. Marotta. Singular perturbations in the McMullen domain. preprint, 2008.

[GMR13] Antonio Garijo, Sebastian M. Marotta, and Elizabeth D. Russell. Singular perturbations in the quadratic family with multiple poles. *Journal of Difference Equations and Applications*, 19(1):1–22, 2013.

[McM88] Curtis T. McMullen. Automorphisms of rational maps. In *Holomorphic functions and moduli, Vol. I*, Mathematical Sciences Research Institute Publications, pages 31–60. Springer-Verlag, New York, NY, 1988.

[McM94] Curtis T. McMullen. *Complex dynamics and renormalization*, volume 135 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, 1994.

[Mil00] John Milnor. On rational maps with two critical points. *Experimental Mathematics*, 9(4):481–522, 2000.

[Mil06] John Milnor. *Dynamics in one complex variable*, volume 160 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, third edition, 2006.

[PT00] Kevin M. Pilgrim and Tan Lei. Rational maps with disconnected Julia set. In *Géométrie complexe et systèmes dynamiques, colloque en l'honneur d'Adrien Douady*, volume 261 of *Astérisque*, pages 349–384. Société Mathématique de France, Paris, 2000.

[Shi87] Mitsuhiro Shishikura. On the quasiconformal surgery of rational functions. *Annales Scientifiques de l'École Normale Supérieure. Quatrième Série*, 20(1):1–29, 1987.